ALMOST COMPLEX MANIFOLDS—III

by R. S. Mishra, F.N.A., *and* Ram Hit, *Department of Mathematics, Banaras Hindu University, Varanasi*

(*Received 4 May 1970*)

In this paper, we have obtained an $*O$-connection D as a linear combination of an arbitrary connection E and some of its properties.

1. Introduction

Let us consider a $2n$-dimensional real manifold M_{2n} of differentiability class C^{r+1}. Let there be defined in M_{2n} a vector-valued linear function F such that

$$\overline{\overline{X}}+X = 0 \qquad \ldots \quad (1.1)$$

for arbitrary vector field X, where

$$\overline{X} \overset{\text{def}}{=} F(X). \qquad \ldots \quad (1.2)$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex manifold.

Agreement 1.1—All the equations which follow hold for arbitrary vector fields X, Y, Z, . . . , etc.

In an almost complex manifold a connection D is said to be an F-connection (Yano 1965) if

$$(D_X F)(Y) = 0. \qquad \ldots \quad (1.3)$$

In view of (1.1) and (1.2), this equation is equivalent to

$$D_X\overline{Y} = \overline{D_X Y}, \quad D_{\overline{X}}\overline{Y} = \overline{D_{\overline{X}} Y}$$

$$-D_{\overline{X}} Y = \overline{D_{\overline{X}}\overline{Y}}, \; -D_X Y = \overline{D_X \overline{Y}}.$$

In an almost complex manifold a connection D is said to be a semi-F-connection (Mishra 1969) if

$$(\text{div } F)(X) = 0 \quad \text{or} \quad (\text{div } F)(\overline{X}) = 0. \qquad \ldots \quad (1.4)$$

In an almost complex manifold a connection D is said to be an M-connection (Mishra 1969) if

$$(D_X F)(Y)+(D_Y F)(X) = 0. \qquad \ldots \quad (1.5)$$

In consequence of (1.1) and (1.2), this equation is equivalent to

$$D_X\overline{Y}+D_Y\overline{X} = \overline{D_X Y}+\overline{D_Y X}$$

$$D_{\overline{X}}\overline{Y}-D_Y X = \overline{D_{\overline{X}} Y}+\overline{D_Y \overline{X}}$$

$$-D_X Y+D_{\overline{Y}}\overline{X} = \overline{D_X \overline{Y}}+\overline{D_{\overline{Y}} X}$$

$$-D_{\overline{X}} Y-D_{\overline{Y}} X = \overline{D_{\overline{X}}\overline{Y}}+\overline{D_{\overline{Y}}\overline{X}}.$$

Definition (1.1)—In an almost complex manifold a connection D will be called an $*O$-connection if

$$(D_X F)(Y)+(D_{\bar{X}} F)(\bar{Y})=0. \quad \ldots \quad (1.6)$$

In an almost complex manifold M_{2n}, let there be defined a Hermite metric g:

$$g(\bar{X}, \bar{Y})=g(X, Y). \quad \ldots \quad (1.7a)$$

Then M_{2n} is said to be an almost Hermite manifold. By virtue of (1.1) and (1.2), eqn. (1.6) is equivalent to

$$\text{(a) } {'F}(X, Y) \stackrel{\text{def}}{=} g(\bar{X}, Y)=-g(X, \bar{Y})$$

$$\text{(b) } {'F}(X, \bar{Y})=-{'F}(\bar{X}, Y)=g(X, Y)$$

$$\text{(c) } {'F}(\bar{X}, \bar{Y})={'F}(X, Y).$$

In an almost Hermite manifold a connection D is said to be an almost F-connection (Mishra 1969) if

$$(D_X{'F})(Y, Z)+(D_Y{'F})(Z, X)+(D_Z{'F})(X, Y)=0. \quad \ldots \quad (1.8)$$

Let S be the torsion tensor of D:

$$S(X, Y)=D_X Y-D_Y X-[X, Y]. \quad \ldots \quad (1.9)$$

Then D is called half-symmetric if

$$S(X, Y)-S(\bar{X}, \bar{Y})=\overline{S(X, \bar{Y})}+\overline{S(\bar{X}, Y)} \quad \ldots \quad (1.10a)$$

or

$$S(X, \bar{Y})+S(\bar{X}, Y)=\overline{S(\bar{X}, \bar{Y})}-\overline{S(X, Y)}. \quad \ldots \quad (1.10b)$$

Let us put

$$T(X) \stackrel{\text{def}}{=} (C_1^1 S)(X). \quad \ldots \quad (1.11a)$$

C_1^1 is the contraction.
Then if

$$nS(X, Y)=XT(Y)-YT(X)+\bar{X}T(\bar{Y})-\bar{Y}T(\bar{X}) \quad \ldots \quad (1.11b)$$

D is said to be semi-symmetric (Mishra 1969).

Let us put

$$U(X) \stackrel{\text{def}}{=} (C_1^1 S)(X). \quad \ldots \quad (1.12a)$$

Then if

$$nS(X, Y)=X\,U(Y)-Y\,U(X)+\bar{X}\,U(\bar{Y})-\bar{Y}\,U(\bar{X}) \quad \ldots \quad (1.12b)$$

D is said to be almost symmetric (Mishra 1969).

We have (Mishra 1969)

$$\begin{aligned} S(X, Y)-S(\bar{X}, \bar{Y})-\overline{S(X, \bar{Y})}-\overline{S(\bar{X}, Y)} &= \{[X, Y]-[\bar{X}, \bar{Y}]-\overline{[\bar{X}, Y]}-\overline{[X, \bar{Y}]}\} \\ &(\alpha-\delta-\phi-\rho-1)+\{\overline{[X, Y]}-\overline{[\bar{X}, \bar{Y}]}+[\bar{X}, Y]+[X, \bar{Y}]\} \\ &(\theta-\sigma+\beta+\gamma)+\{s(X, Y)-s(\bar{X}, \bar{Y})-\overline{s(X, \bar{Y})}-\overline{s(\bar{X}, Y)}\} \\ &(\alpha-\delta-\phi-\rho)+\{\overline{s(X, Y)}-\overline{s(\bar{X}, \bar{Y})}+s(\bar{X}, Y)+s(X, \bar{Y})\} \\ &(\theta-\sigma+\beta+\gamma) \quad \ldots \quad (1.13) \end{aligned}$$

2. *O-Connection

Theorem (2.1)—Let E be an arbitrary connection in M_{2n}. Then the connection D defined by

$$D_X Y = (-\phi+\rho-\delta)E_X Y+(\sigma+\gamma+\theta)E_X \bar{Y}+\gamma E_{\bar{X}} Y+\delta E_{\bar{X}} \bar{Y} +\theta\overline{E_X Y}+\phi\overline{E_X \bar{Y}}+\rho\overline{E_{\bar{X}} Y}+\sigma\overline{E_{\bar{X}} \bar{Y}} \quad \ldots \quad (2.1a)$$

is an *O-connection.

Proof: Let us put

$$D_X Y \overset{\text{def}}{=} \alpha E_X Y+\beta E_X \bar{Y}+\gamma E_{\bar{X}} Y+\delta E_{\bar{X}} \bar{Y}+\theta\overline{E_X Y}+\phi\overline{E_X \bar{Y}}+\rho\overline{E_{\bar{X}} Y}+\sigma\overline{E_{\bar{X}} \bar{Y}}. \quad \ldots \quad (2.2)$$

Barring this equation throughout, we get

$$\overline{D_X Y} = \alpha\overline{E_X Y}+\beta\overline{E_X \bar{Y}}+\gamma\overline{E_{\bar{X}} Y}+\delta\overline{E_{\bar{X}} \bar{Y}}-\theta E_X Y-\phi E_X \bar{Y}-\rho E_{\bar{X}} Y-\sigma E_{\bar{X}} \bar{Y}. \quad \ldots \quad (2.3a)$$

From (1.1) and (2.3a), we have

$$\overline{D_X Y}+\overline{D_{\bar{X}} \bar{Y}} = (\alpha+\delta)(\overline{E_X Y}+\overline{E_{\bar{X}} \bar{Y}})+(\beta-\gamma)(\overline{E_X \bar{Y}}-\overline{E_{\bar{X}} Y}) -(\theta+\sigma)(E_X Y+E_{\bar{X}} \bar{Y})-(\phi-\rho)(E_X \bar{Y}-E_{\bar{X}} Y). \quad (2.3b)$$

Barring Y in (2.2), we get

$$(D_X F)(Y)+\overline{D_X Y} = \alpha E_X \bar{Y}-\beta E_X Y+\gamma E_{\bar{X}} \bar{Y}-\delta E_{\bar{X}} Y+\theta\overline{E_X \bar{Y}} -\phi\overline{E_X Y}+\rho\overline{E_{\bar{X}} \bar{Y}}-\sigma\overline{E_{\bar{X}} Y}. \quad \ldots \quad (2.4a)$$

From (1.1), (1.6) and (2.4a), we have

$$\overline{D_X Y}+\overline{D_{\bar{X}} \bar{Y}} = (\alpha+\delta)(E_X \bar{Y}-E_{\bar{X}} Y)-(\beta-\gamma)(E_X Y+E_{\bar{X}} \bar{Y}) +(\theta+\sigma)(\overline{E_X \bar{Y}}-\overline{E_{\bar{X}} Y})-(\phi-\rho)(\overline{E_X Y}+\overline{E_{\bar{X}} \bar{Y}}). \quad (2.4b)$$

Comparing (2.3b) and (2.4b), we get

$$\alpha = -\phi+\rho-\delta, \quad \beta = \sigma+\gamma+\theta. \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

Substituting from the above equations in (2.2), we obtain (2.1).

Corollary 2.1—The equation (2.1a) is equivalent to

$$D_X \bar{Y} = (-\phi+\rho-\delta)E_X \bar{Y}-(\sigma+\gamma+\theta)E_X Y+\gamma E_{\bar{X}} \bar{Y}-\delta E_{\bar{X}} Y +\theta\overline{E_X \bar{Y}}-\phi\overline{E_X Y}+\rho\overline{E_{\bar{X}} \bar{Y}}-\sigma\overline{E_{\bar{X}} Y} \quad \ldots \quad (2.1b)$$

$$D_{\bar{X}} Y = (-\phi+\rho-\delta)E_{\bar{X}} Y+(\sigma+\gamma+\theta)E_{\bar{X}} \bar{Y}-\gamma E_X Y -\delta E_X \bar{Y}+\theta\overline{E_{\bar{X}} Y}+\phi\overline{E_{\bar{X}} \bar{Y}}-\rho\overline{E_X Y}-\sigma\overline{E_X \bar{Y}} \quad \ldots \quad (2.1c)$$

$$D_{\bar{X}} \bar{Y} = (-\phi+\rho-\delta)E_{\bar{X}} \bar{Y}-(\sigma+\gamma+\theta)E_{\bar{X}} Y-\gamma E_X \bar{Y}+\delta E_X Y +\theta\overline{E_{\bar{X}} \bar{Y}}-\phi\overline{E_{\bar{X}} Y}-\rho\overline{E_X \bar{Y}}+\sigma\overline{E_X Y} \quad \ldots \quad (2.1d)$$

$$\overline{D_X Y} = (-\phi+\rho-\delta)\overline{E_X Y}+(\sigma+\gamma+\theta)\overline{E_X \bar{Y}}+\gamma\overline{E_{\bar{X}} Y}+\delta\overline{E_{\bar{X}} \bar{Y}} -\theta E_X Y-\phi E_X \bar{Y}-\rho E_{\bar{X}} Y-\sigma E_{\bar{X}} \bar{Y} \quad \ldots \quad (2.1e)$$

$$\overline{D_X\overline{Y}} = (-\phi+\rho-\delta)\overline{E_X\overline{Y}}-(\sigma+\gamma+\theta)\overline{E_X Y}+\gamma\overline{E_{\overline{X}}\overline{Y}}-\delta\overline{E_{\overline{X}}Y}$$
$$-\theta E_X\overline{Y}+\phi E_X Y-\rho E_{\overline{X}}\overline{Y}+\sigma E_{\overline{X}}Y \quad \ldots \quad \ldots \quad (2.1f)$$

$$\overline{D_{\overline{X}}Y} = (-\phi+\rho-\delta)\overline{E_{\overline{X}}Y}+(\sigma+\gamma+\theta)\overline{E_{\overline{X}}\overline{Y}}-\gamma\overline{E_X Y}-\delta\overline{E_X\overline{Y}}$$
$$-\theta E_{\overline{X}}Y-\phi E_{\overline{X}}\overline{Y}+\rho E_X Y+\sigma E_X\overline{Y} \quad \ldots \quad \ldots \quad (2.1g)$$

$$\overline{D_{\overline{X}}\overline{Y}} = (-\phi+\rho-\delta)\overline{E_{\overline{X}}\overline{Y}}-(\sigma+\gamma+\theta)\overline{E_{\overline{X}}Y}-\gamma\overline{E_X\overline{Y}}+\delta\overline{E_X Y}$$
$$-\theta E_{\overline{X}}\overline{Y}+\phi E_{\overline{X}}Y+\rho E_X\overline{Y}-\sigma E_X Y. \quad \ldots \quad \ldots \quad (2.1h)$$

PROOF: Barring different vectors in (2.1a) and using (1.1) and (1.2), we obtain (2.1b)–(2.1h).

Theorem 2.2—An $*O$-connection is a semi-F-connection

PROOF: From (2.3a) and (2.4a), we have

$$(D_X F)(Y) = (\alpha+\phi)(E_X F)(Y)+(\theta-\beta)\overline{(E_X F)(\overline{Y})}+(\gamma+\sigma)(E_{\overline{X}}F)(Y)$$
$$+(\rho-\delta)\overline{(E_{\overline{X}}F)(\overline{Y})}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

Contracting this equation, we get

$$(\mathrm{div}_D\, F)(Y) = (\alpha+\delta+\phi-\rho)(\mathrm{div}_E\, F)(Y)+(\beta-\sigma-\gamma-\theta)(\mathrm{div}_E\, F)(\overline{Y}) \qquad (2.7)$$

where div_D refers to the connection D and div_E refers to the connection E. Using (2.5) in (2.7), we get

$$(\mathrm{div}\, F)(X) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

which proves the statement.

Theorem 2.3—Let E be an arbitrary connection given by (2.2). The condition for an $*O$-connection to be an F-connection or an M-connection or an almost F-connection is

$$\alpha+\phi = 0, \quad \beta = \theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9a)$$

or

$$\rho = \delta, \quad \sigma+\gamma = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9b)$$

PROOF: The connection D is the most general F-connection if (Mishra 1967)

$$\alpha+\phi = 0, \quad \beta = \theta, \quad \gamma+\sigma = 0, \quad \rho = \delta. \quad \ldots \quad \ldots \quad (2.10)$$

The statement follows from (2.5) and (2.10).

Theorem 2.4—Let connections D and E be related by (2.2). If any two of the following properties hold, the third also holds:

(i) connection D is semi-symmetric,

(ii) connection E is almost symmetric,

(iii) $[\overline{X}, \overline{Y}]-[X, Y]+\overline{[X, \overline{Y}]}+\overline{[\overline{X}, Y]} = 0$.

PROOF: The statement follows from (1.11), (1.12) and (1.13).

Theorem 2.5—Let connections D and E be related by (2.2). If any two of the following properties hold, the third also holds:

(i) connection D is half-symmetric,

(ii) connection E is almost symmetric,

(iii) $[\bar{X}, \bar{Y}]-[X, Y]+\overline{[X, \bar{Y}]}+\overline{[\bar{X}, Y]} = 0.$

PROOF: The statement follows from (1.10), (1.12) and (1.13).

REFERENCES

Mishra, R. S. (1967). On almost complex spaces. General F-connection. *Prog. Math.*, **1**, No. 2, 31–35.

——— (1969). On almost complex manifold. *Tensor* (N.S.), **20**, 361–66.

Yano, K. (1965). Differential Geometry on Complex and Almost Complex Spaces. Pergamon Press, New York.