STRONG PSEUDO-CONVEX PROGRAMMING

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The duality theory for strong pseudo-convex (concave) programming problems is studied and its application to the tensor products of non-linear programmes is presented.

1. INTRODUCTION

Mangasarian (1965) introduced pseudo-convex functions and among other things studied the duality theory for programming problems with pseudo-convex objective function and quasi-convex constraints. He established the converse duality theorem (under some appropriate conditions) and observed that the direct duality theorem does not hold good in general for this class of programmes. Bector (1968) and Kantiswarup (1966) also studied duality relationships for fractional and indefinite programming problems and proved results similar to Mangasarian (1965).

In this paper, we consider a subclass of the programming problems considered by Mangasarian (1965) (called strong pseudo-convex programming problems) and study duality aspect for such programmes. The main result proved is the direct duality theorem, the converse being true because of Mangasarian (1965). Mond's (1965) results for the tensor products of convex (concave) programmes are extended to this class of programmes.

2. PRELIMINARIES

Let R^n be the Euclidean space of dimensions n and D a convex subset in R^n. Let C^1 denote the class of all continuous functions $f : D \rightarrow R$ such that every first order partial derivative of f exists and is continuous in D. Also let '∇' denote the usual gradient operator. Now we have the following definitions.

Definition 1—(Strong pseudo-convex function) $f : D \rightarrow R$ is said to be strong pseudo-convex if $f \in C^1$ and for every pair of points $x_1, x_2 \in P$, a real valued function $K(x_1, x_2) > 0$ such that

$$(x_1 - x_2)T \nabla f(x_2) \leqslant K(x_1, x_2)[f(x_1) - f(x_2)].$$

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Definition 2—(Strong pseudo-concave function) $f: D \to R$ is said to be strong pseudo-concave if and only if $-f$ is strong pseudo-convex.

For other definitions of convex-like functions, we shall refer to Bector (1969) and Mangasarian (1965).

Bector (1969) also studied the nature of quotients and products of convex-like functions and established the following results (given in the form of Lemmas) which we shall use in the latter part of the paper.

Lemma (2.1)—If (i) $f: D \to R$ is non-negative convex (concave), (ii) $g: D \to R$ is strictly positive concave (convex) and (iii) $f, g \in C^1$, then the function $h: D \to R$ given by $h(x) = [f(x)/g(x)]$ is strong pseudo-convex (concave).

Lemma (2.2)—If (i) $f: D \to R$ is non-negative concave, (ii) $g: D \to R$ is strictly positive concave and (iii) $f, g \in C^1$, then the function $h: D \to R$, given by $h(x) = [f(x)\, g(x)]$ is strong pseudo-concave.

It can be noted that the class of strong pseudo-convex (concave) functions lies in between the class of pseudo-convex (concave) and convex (concave) functions and hence any result which is true for pseudo-convex (concave) functions will remain true for strong pseudo-convex (concave) functions also. The motivation for studying this class of functions becomes clear from Lemmas (2.1) and (2.2).

3. Statement of the Problem

Let $I = \{1, 2, \ldots, m\}$ be the index set and F, $g_i (i \in I)$ be functions belonging to the class C^1. The primal and dual problems, formulated by Mangasarian (1965) and Wolfe (1961) are as follows:

$$\text{Primal Problem } (P): \quad \left.\begin{array}{l} \text{minimize } F(x) \\ \text{subject to} \\ \qquad g_i(x) \leqslant 0, \ (i \in I) \end{array}\right\} \quad \ldots \quad \ldots \quad (3.1)$$

$$\text{Dual Problem } (D): \quad \left.\begin{array}{ll} \text{maximize} & F(x) + \sum_{i \in I} \lambda_i g_i(x) \\ \text{subject} & \nabla F(x) + \sum_{i \in I} \lambda_i \nabla g_i(x) = 0 \\ & \lambda_i \geqslant 0, \ i \in I \end{array}\right\}. \quad \ldots \quad (3.2)$$

Mangasarian (1965) established the converse duality theorem for the case when F is pseudo-convex and $g_i (i \in I)$ are quasi-convex. Also, he gave a counter example to show that the direct duality theorem does not hold good in general for this class of programmes. In the following we shall assume that the functions F, $g_i (i \in I)$ are strong pseudo-convex and Kuhn-Tucker (1951) constraint qualifications are satisfied for the primal programme (P).

4. Duality

The converse duality theorem is clearly true for the programmes (P) and (D), under Mangasarian's (1965) hypothesis. To study direct duality aspect, we need the following Lemma.

Lemma (4.1)—The function h given by $h = F + \sum_{i \in I} \lambda_i g_i$ is strong pseudo-convex for $\lambda_i \geqslant 0$ $(i \in I)$.

The proof is immediate from the definition of strong pseudo-convexity.

Theorem (4.1): (*Weak Duality Theorem*)—If x_0 is feasible for the primal programme (P) and $(x_1, \lambda_1, \lambda_2, \ldots, \lambda_m)$ is feasible for the dual programme (D) then:

$$F(x_1) + \sum_{i \in I} \lambda_i g_i(x_1) \leqslant F(x_0) \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

PROOF: $(x_1, \lambda_1, \lambda_2, \ldots, \lambda_m)$ feasible for programme (D)

$$\Rightarrow \nabla F(x_1) + \sum_{i \in I} \lambda_i \nabla g_i(x_1) = 0$$

$$\Rightarrow (x_0 - x_1)^T \left[\nabla F(x_1) + \sum_{i \in I} \lambda_i \nabla g_i(x_1) \right] = 0$$

$$\Rightarrow \left[F(x_0) + \sum_{i \in I} \lambda_i g_i(x_0) \right] \geqslant \left[F(x_1) + \sum_{i \in I} \lambda_i g_i(x_1) \right]$$

(by Lemma (4.1))

$$\Rightarrow F(x_0) \geqslant F(x_1) + \sum_{i \in I} \lambda_i g_i(x_1) - \sum_{i \in I} \lambda_i g_i(x_0)$$

$$\Rightarrow F(x_0) \geqslant F(x_1) + \sum_{i \in I} \lambda_i g_i(x_1)$$

where the last implication follows from the fact that x_0 is primal feasible and $\lambda_i (i \in I)$ are non-negative scalars.

Corollary (4.1)—If equality holds at (4.1) then x_0 and $(x_1, \lambda_1, \lambda_2, \ldots, \lambda_m)$ are optimal for the respective programmes (P) and (D).

Theorem (4.2): (*Direct Duality Theorem*—If x_0 is an optimal solution of the programme (P) then there exist scalars $\lambda_i (i \in I)$ such that $(x_0, \lambda_1, \lambda_2, \ldots, \lambda_m)$ is optimal for the programme (D) and the two extreme values are equal.

PROOF: Since x_0 is optimal for the programme (P), there exist scalars $\lambda_i^0 (i \in I)$ (Mangasarian (1965), Theorem 1) satisfying:

$$\left.\begin{aligned} \nabla F(x_0) \sum_{i \in I} \lambda_i^0 \nabla g_i(x_0) &= 0 \\ \sum_{i \in I} \lambda_i^0 g_i(x_0) &= 0 \\ g_i(x_0) &\leqslant 0 \ (i \in I) \\ \lambda_i^0 &\geqslant 0 \ (i \in I) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

Hence $(x_0, \lambda_1^0, \lambda_2^0, \ldots, \lambda_m^0)$ is dual feasible and theorem follows from the fact that

$$F(x_0) = F(x_0) + \sum_{i \in I} \lambda_i^0 g_i(x_0)$$

[Corollary (4.1)].

Remark: In view of Lemmas (2.1) and (2.2), we see that the direct duality theorem holds good for the following type of programmes which are more general than the usual convex programmes.

(a) *Fractional Programme*—The problem is to find an $x \in R^n$ such that

$$\phi(x) = [f(x)/g(x)] \text{ is minimum}$$

subject to

$$h(x) \leqslant 0 \ (i \in I)$$

where f is a non-negative convex function and g is a strictly positive convex function. The constraint functions $h_i(i \in I)$ are strong pseudo-convex and all functions f, g, $h_i(i \in I)$ are members of the class C^1.

(b) *Indefinite Programme*—The problem is to find an $x \in R^n$ such that

$$(x) = [f(x)g(x)] \text{ is maximum}$$

subject to

$$h_i(x) < 0, \ i \in I$$

where f and g are concave functions $\in C^1$ such that $f \geqslant 0$ and $g > 0$. The constraints functions $h_i(i \in I)$ are taken same as in the case (a). In this case we have to make necessary modifications in the formulation of the dual problem as the primal objective function is to be maximized.

5. Application

In this section we shall give certain applications of the results developed in section 3 to the study of large structured mathematical programmes. Mond (1965) considered the direct sums and tensor products of non-linear programmes (in particular convex programmes) and obtained conditions for obtaining an optimal vector of the composed programme from optimal vectors of the component programmes. It can be noted that Mond's (1965) results depend heavily on the duality aspect of these programmes and hence cannot be taken over to pseudo-convex programmes in general. In what follows we shall show that because of theorem (4.2), Mond's results are extendable to the strong pseudo-convex programmes.

Let us consider a finite sequence (N) of mathematical programmes M^i ($i = 1, 2, \ldots, N$), given by

$$M^i: \left.\begin{array}{l} \text{minimize } f^i(\underline{x}^i) \\ \text{subject to } A^i\underline{x}^i = \underline{b}^i \\ \underline{x}^i \geqslant 0 \end{array}\right\} \quad \ldots \quad (5.1)$$

and define the tensor product programme $\otimes M^i$ as follows:

$$\otimes M^i: \left.\begin{array}{l} \text{minimize } g(\otimes \underline{x}^i) \\ \text{subject to } (\otimes A^i)(\otimes \underline{x}^i) = (\otimes \underline{b}^i) \\ \otimes \underline{x}^i \geqslant 0 \end{array}\right\} \quad \ldots \quad (5.2)$$

Denoting $V^i = \{\underline{x}^i / A^i\underline{x}^i = \underline{b}^i\}$ and $(M^i)^*$ as the programme dual to M^i, we have the following theorem:

Theorem (5.1)—A tensor product $\otimes \underline{x}^i$ of optimal vectors $\underline{x}^i$ is optimal for $\otimes M^i$, is $\nabla g(\otimes \underline{x}^i) = \otimes[\nabla f^i(\underline{x}^i)]$ for all $\underline{x}^i \in V^i$, and there exist a family of vectors $\underline{y}^i$ such that $(\underline{y}^i, \underline{x}^i)$ is optimal for (M^{i*}) and such that $(\otimes \underline{y}^i, \otimes \underline{x}^i)$ is feasible solution of $(x\ M^i)^*$.

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