ROTATIONAL OSCILLATIONS OF A SPHERE ABOUT A FIXED DIAMETER IN A RIVLIN-ERICKSEN FLUID

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The flow generated by the rotational oscillations of a sphere in an infinitely extending fluid has been first studied by Carrier and Di Prima (1956). In recent years much work has been done for the corresponding problem for non-Newtonian fluids restricted to the steady component of the secondary flow alone. In the present paper the secondary flow induced in a visco-elastic fluid governed by Rivlin-Ericksen, taking into account the fluctuating component also, has been studied in detail and we have arrived at some interesting results.

1. Introduction

In recent years much work has been done on the problem of the rotational oscillations of a sphere about a fixed diameter in an infinitely extending non-Newtonian fluid (Frater 1964, Bhatnagar 1965, Bhatnagar (P. L.) 1968(a, b) and 1970, Rajagopalan and Bhatnagar 1967, Bhatnagar and Rajagopalan 1967). All these studies, however, were restricted to the investigation of the steady component of the secondary flow alone. This, in general, does not represent the entire secondary flow. This paper presents the study of both the steady and fluctuating components of the secondary flow induced in a viscoelastic fluid governed by Rivlin-Ericksen constitutive equation by a sphere performing rotational oscillations of small angular amplitudes about a fixed diameter. Carrier and Di Prima (1956) first initiated investigations in this field by considering a Newtonian viscous fluid. In short, we may mention that this paper reports for the first time the study of the fluctuating components.

We have made the assumption that the angular amplitude of oscillation Ω is small so that the non-linear equations may greatly be simplified. This allows us to expand all flow variables in powers of Ω and the first and the second order approximations give us the first and the second order motion which are commonly referred to as primary and secondary motions. The specific purpose of this study is to investigate the unsteady part of the second order motion.

2. Formulation of the Problem

We have considered a spherical polar system (r, θ, ϕ) with the origin at the centre of the sphere of radius a. The sphere performs rotational oscillations with angular amplitude Ω and frequency n about the diameter $\theta = 0$

in an infinitely extending Rivlin-Ericksen fluid. If u, v, w are velocity components in the increasing directions of r, θ, ϕ then the boundary conditions to be specified are:

$$\left.\begin{array}{llll} u=0, & v=0, & w=a\Omega \sin\theta e^{int} & \text{on } r=a \\ u\to 0, & v\to 0, & w\to 0 & \text{as } r\to\infty \end{array}\right\} \quad (2.1)$$

The continuity, momentum and constitutive equations are as follows:

$$u_{i,i}=0 \quad (2.2)$$

$$\rho\left(\frac{\partial u_i}{\partial t}+u^m u_{i,m}\right)=T_{ij,j} \quad (2.3)$$

$$T_{ij}=-p\delta ij+p_{ij} \quad (2.4)$$

$$p_{ij}=2\phi_1 E_{ij}+\phi_2 D_{ij}+4\phi_3 E_i{}^m E_{mj} \quad (2.5)$$

where the symbols involved have the usual significance and the suffix following a comma denotes covariant differentiation.

We introduce the dimensionless quantities through the following relations:

$$\left.\begin{array}{l} r=ar', \quad t=n^{-1}t', \quad u=anu', \quad v=anv' \\ w=anw', \quad p_{ik}=n\phi_1 p'_{ik}, \quad p=\rho a^2n^2p' \\ R=\dfrac{\rho a^2 n}{\phi_1}=\text{Reynolds number}, \quad \Omega=n\Omega' \\ K=\dfrac{n\phi_2}{\phi_1} \text{ and } S=\dfrac{n\phi_3}{\phi_1} \end{array}\right\} \quad (2.6)$$

Henceforth the prime denoting dimensionless quantities is dropped for simplicity and the real parts are to be understood wherever complete expressions are quoted for physical quantities.

The boundary conditions (2.1) reduce to the following form:

$$\left.\begin{array}{llll} u=0, & v=0, & w=\Omega \sin\theta e^{it} & \text{on } r=1 \\ u\to 0, & v\to 0, & w\to 0 & \text{as } r\to\infty \end{array}\right\} \quad (2.7)$$

Now we expand the variables as a power series in Ω assuming the parameter Ω to be small in the following form:

$$\left.\begin{array}{l} w=\Omega f(r)\sin\theta e^{it}+\Omega^3 f_1(r,\theta,t)+\dots \\ u=\Omega^2 g(r,\theta,t)+\dots \\ v=\Omega^2 h(r,\theta,t)+\dots \\ p_{\phi r}=\Omega F(r)\sin\theta e^{it}+\Omega^3 F_1(r,\theta,t)+\dots \\ p_{rr}=\Omega^2 G(r,\theta,t)+\dots \\ p_{\theta\theta}=\Omega^2 H(r,\theta,t)+\dots \\ p_{\phi\phi}=\Omega^2 Q(r,\theta,t)+\dots \\ p_{r\theta}=\Omega^2 L(r,\theta,t)+\dots \\ p=\Omega^2 N(r,\theta,t)+\dots \end{array}\right\} \quad (2.8)$$

The first and second order motions are generated by the equations:

$$F = (1+iK)r\frac{d}{dr}\left(\frac{f}{r}\right) \qquad \ldots \quad (2.9)$$

$$if = \frac{1}{R}\left[\frac{1}{r^3}\frac{d}{dr}(r^3F)\right] \qquad \ldots \quad (2.10)$$

$$G = 2\left(1+K\frac{\partial}{\partial t}\right)\frac{\partial g}{\partial r}+(2K+S)r^2\sin^2\theta\left\{\frac{d}{dr}\left(\frac{f}{r}\right)e^{it}\right\}^2 \qquad \ldots \quad (2.11)$$

$$H = 2\left(1+K\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial h}{\partial \theta}+\frac{g}{r}\right) \qquad \ldots \quad (2.12)$$

$$M = 2\left(1+K\frac{\partial}{\partial t}\right)\left(\frac{g}{r}+\frac{h\cot\theta}{r}\right)+Sr^2\sin^2\theta\left\{\frac{d}{dr}\left(\frac{f}{r}\right)e^{it}\right\}^2 \qquad \ldots \quad (2.13)$$

$$L = \left(1+K\frac{\partial}{\partial t}\right)\left[\frac{1}{r}\frac{\partial g}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{h}{r}\right)\right] \qquad \ldots \quad (2.14)$$

$$\frac{\partial g}{\partial t}-\frac{\sin^2\theta}{r}(fe^{it})^2 = -\frac{\partial N}{\partial r}+\frac{1}{R}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2G)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(L\sin\theta-\frac{H+M}{r}\right)\right] \qquad \ldots \quad (2.15)$$

$$\frac{\partial h}{\partial t}-\frac{\sin 2\theta}{2r}(fe^{it})^2 = -\frac{1}{r}\frac{\partial N}{\partial \theta}+\frac{1}{R}\left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3L)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H\sin\theta)-\frac{M\cot\theta}{r}\right], \qquad \ldots \quad (2.16)$$

$$\frac{\partial}{\partial r}(r^2g\sin\theta)+\frac{\partial}{\partial\theta}(rh\sin\theta) = 0. \qquad \ldots \quad (2.17)$$

With boundary conditions:

$$\left.\begin{array}{llll} f=1, & g=0, & h=0. & \text{on } r=1, \\ f\to 0, & g\to 0 & h\to 0 & \text{as } r\to\infty \end{array}\right] \qquad \ldots \quad (2.18)$$

where $R = \dfrac{a^2n\rho}{\phi_1}$ is the Reynolds number for the flow and $K = \dfrac{n\phi_2}{\phi_1}$, $S = n\dfrac{\phi_3}{\phi_1}$ are the non-dimensional parameters specifying the viscoelasticity and cross-viscosity of the fluid.

3. Solution of the Equations

The first order motion has been studied in detail by Frater (1964), Bhatnagar (1965), Bhatnagar and Rajagopalan (1967). The motion lies entirely in planes perpendicular to the axis of oscillation and the streamlines are circles with the centres on the axis. This motion is purely harmonic with the same period as that of the oscillating boundary. It is dependent on the viscous and the elastic parameters of the fluid, unlike the case of rotatory motion where the elastic parameters affect only the second order motion. The non-vanishing components of velocity w of the first order motion can be written in the form

$$w = (w_N + w_{NN})e^{it} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where w_N denotes the Newtonian and w_{NN} denotes the non-Newtonian contribution to the velocity. w_N is the same for all fluids while w_{NN} specifies the characteristic elastic response of each particular fluid (Bhatnagar 1968*a*, *b* and 1970).

The second order motion is purely a meridional one and lies entirely in planes containing the axis of oscillation. This motion is determined by the velocity components u and v, which are smaller order of magnitude in our scheme of perturbation than the component w, occurring in the first order motion. The equations (2.11) to (2.18) determine the second order motion. From the terms of the types $\left\{\frac{d}{dr}(f/r)e^{it}\right\}^2$ and $\{fe^{it}\}^2$ in these equations, we can make certain deductions about the dependence of the quantities on time. We have taken the real part of the complex expressions for physical quantities. According to our notations, fe^{it} for instance stands for the real part of fe^{it} and if z_1 and z_2 are any two complex quantities then

$$Rl(z_1)Rl(z_2) = 1/2[Rl(z_1\bar{z}_2) + Rl(z_1z_2)] \qquad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

so that

$$(fe^{it})(fe^{it}) = 1/2[f\bar{f} + f^2e^{2it}]. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

The quantities are involved in eqns. (2.11)–(2.16).

We infer that the second order motion can be expressed as the sum of two terms, one of which is time independent (steady) and the other containing a factor of the type exp $(2it)$, i.e. a fluctuating component with twice the frequency of the oscillating boundary. Thus we can separate the steady and time-dependent parts by setting

$$\left.\begin{aligned}
g(r, \theta, t) &= g_1(r, \theta) + g_2(r, \theta)e^{2it} \\
h(r, \theta, t) &= h_1(r, \theta) + h_2(r, \theta)e^{2it} \\
f(r, \theta, t) &= f_1(r, \theta) + f_2(r, \theta)e^{2it} \\
G(r, \theta, t) &= G_1(r, \theta) + G_2(r, \theta)e^{2it} \\
H(r, \theta, t) &= H_1(r, \theta) + H_2(r, \theta)e^{2it} \\
Q(r, \theta, t) &= Q_1(r, \theta) + Q_2(r, \theta)e^{2it} \\
M(r, \theta, t) &= M_1(r, \theta) + M_2(r, \theta)e^{2it} \\
F(r, \theta, t) &= F_1(r, \theta) + F_2(r, \theta)e^{2it} \\
L(r, \theta, t) &= L_1(r, \theta) + L_2(r, \theta)e^{2it} \\
N(r, \theta, t) &= N_1(r, \theta) + N_2(r, \theta)e^{2it}
\end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (3.4)$$

Substituting these into eqns. (2.11)–(2.17) we get two systems of equations, one for each component of second order motion. The steady component has been evaluated previously (Frater 1964, Bhatnagar 1965, Rajagopalan and Bhatnagar 1967). If we define the stream function ψ_s of the steady component by the relation

$$\left.\begin{aligned} g_1 &= \frac{1}{r^2 \sin\theta}\frac{\partial\psi_s}{\partial\theta} \\ h_1 &= \frac{-1}{r\sin\theta}\frac{\partial\psi_s}{\partial r} \end{aligned}\right\} \qquad (3.5)$$

then

$$\psi_s = \frac{2R\sin^2\theta\cos\theta}{b^2+q^2+2b+1}\left[\left\{\frac{B_1}{r^3}+\frac{B_2}{r^2}+\frac{B_3}{r}+B_4+B_5r+B_6r^2+B_7r^3+B_8r^4\right\}\exp(-2b(r-1))\right.$$

$$\left.+(B_9r^3+B_{10}r^5)\int_r^\infty \frac{\exp(-2b(r-1))}{r}\,dr+C_1+\frac{C_2}{r^2}\right] \qquad (3.6)$$

where

$$JR^{1/2} = b+iq, \quad J^2 = \frac{i}{1+iK} \qquad (3.7)$$

$$\left.\begin{aligned}
B_1 &= 1/4(K+S) \\
B_2 &= -\frac{R}{160b^3}(5b^2+q^2)+\frac{2K+S}{560b^3}(35b^4+42b^2q^2+3q^4) \\
&\quad +\frac{S}{160b^3}(5b^4+14b^2q^2+q^4) \\
B_3 &= \frac{q^2}{560b^2}[-7R+2(2K+S)(7b^2+3q^2)+7S(-b^2+q^2)] \\
B_4 &= -\frac{q^2}{840b}[-7R+2(2K+S)(7b^2-q^2)+7S(-b^2+q^2)] \\
B_5 &= \frac{+q^2}{840}[-7R+2(2K+S)(7b^2-4q^2)+7S(-b^2+q^2)] \\
B_6 &= \frac{-q^2b}{420}[-7R+2(2K+S)(7b^2-6q^2)+7S(-b^2+q^2)] \\
B_7 &= \frac{q^4b^2}{210}(2K+S) \\
B_8 &= -\frac{q^4b^3}{105}(2K+S) \\
B_9 &= -\frac{b^2q^2}{30}[-R+2(2K+S)(b^2-q^2)+S(-b^2+q^2)] \\
B_{10} &= -\frac{2b^4q^4}{105}(2K+S)
\end{aligned}\right\} \qquad (3.8)$$

and

C_1 and C_2 are arbitrary constants to be determined from the boundary conditions

$$\psi_s = \frac{\partial \psi_s}{\partial r} = 0 \quad \text{on } r = 1. \qquad \text{(3.9)}$$

For the fluctuating component of the second order motion, we define the stream function $\psi_f(r, \theta)\, e^{2it}$ by the relation

$$\begin{aligned} g_2 &= \frac{1}{r^2 \sin\theta} \frac{\partial \psi_f}{\partial \theta} \\ h_2 &= -\frac{1}{r \sin\theta} \frac{\partial \psi_f}{\partial r}. \end{aligned} \qquad \text{(3.10)}$$

If we put

$$\psi_f(r, \theta) = \chi_f(r) \sin^2\theta \cos\theta \qquad \text{(3.11)}$$

then $\chi_f(r)$ is found to satisfy the equation

$$L^{*2}\chi_f - m^2 R L^* \chi_f = \xi(r) \qquad \text{(3.12)}$$

where

$$\xi(r) = \frac{1}{1+2iK}\left[R\left(w\frac{dw}{dr} - \frac{w^2}{r}\right) - (2K+S)\left\{r^2 \frac{d}{dr}\left(\frac{d}{dr}\left(\frac{w}{r}\right)\right) + 4r\left(\frac{d}{dr}\left(\frac{w}{r}\right)\right)^2\right\} - \frac{S}{2} r^2 \frac{d}{dr}\left\{\frac{d}{dr}\left(\frac{w}{r}\right)\right\}^2\right] \qquad \text{(3.13)}$$

$$w(r) = \frac{1+JR^{1/2}r}{(1+JR^{1/2})r^2} \exp\left(-JR^{1/2}(r-1)\right) \qquad \text{(3.14)}$$

$$L^* = \frac{d^2}{dr^2} - \frac{6}{r^2} \qquad \text{(3.15)}$$

$$m^2 = \frac{2i}{1+2iK} \qquad \text{(3.16)}$$

and χ_f satisfies the boundary conditions

$$\left.\begin{aligned} &\chi_f = \chi'_f = 0 && \text{on } r = 1, \\ &\chi_f \text{ and } \chi'_f = 0 && \text{as } r \to \infty \end{aligned}\right]. \qquad \text{(3.17)}$$

The solution of eqn. (2.18) can be written in the form by the method of variation of parameters

$$\begin{aligned} \chi_f = {} & \frac{D_1}{r^2} + D_2 r^3 + D_3\left(1 - \frac{3}{mR^{1/2}r} + \frac{3}{m^2Rr^2}\right)\exp\left(mRr^{1/2}\right) \\ & + D_4\left(1 + \frac{3}{mR^{1/2}r} + \frac{3}{m^2Rr^2}\right)\exp\left(-mRr^{1/2}\right) \\ & - \frac{1}{5m^2Rr^2}\int_r^\infty r^3\xi(r)\,dr + \frac{r^3}{5m^2R}\int_r^\infty \frac{\xi(r)}{r^2}\,dr \\ & - \frac{1}{2m^3R^{3/2}} l_1(r)\int_r^\infty \xi(r) l_2(r)\,dr + \frac{l_2(r)}{2m^3R^{3/2}}\int_r^\infty \xi(r) l_1(r)\,dr \end{aligned} \qquad \text{(3.18)}$$

where D_1, D_2, D_3 and D_4 are the arbitrary constants to be determined from boundary conditions (3.17) and

$$\left.\begin{aligned} l_1(r) &= \left(1-\frac{3}{mR^{1/2}r}+\frac{3}{m^2Rr^2}\right)\exp\,(mR^{1/2}r) \\ l_2(r) &= \left(1+\frac{3}{mR^{1/2}r}+\frac{3}{m^2Rr^2}\right)\exp\,(-mR^{1/2}r) \end{aligned}\right\} \quad \ldots \quad (3.19)$$

Hence we choose $D_2 = D_3 = 0$, where we associate the positive sign with the real part of $mR^{1/2}$. Substituting the value of $\xi(r)$ in (3.18) and simplifying, we have

$$\begin{aligned} \chi_f &= \frac{D_1}{r^2}+D_4A(m, R, r)\exp\,(-mR^{1/2}r) \\ &-\frac{\exp\,[-2JR^{1/2}(r-1)]}{5m^2R(1+2iK)(1+JR^{1/2})^2}[-RF^*(J, R, r) \\ &+2(2K+S)G^*(J, R, r)+SH^*(J, R, r)] \\ &-\frac{\exp\,[-2JR^{1/2}(r-1)]}{2m^5R^{5/2}(1+2iK)(1+JR^{1/2})^2}[A(-m, R, r)][-RM^*(m, J, R, r) \\ &+2(2K+S)N^*(m, J, R, r)+SP^*(m, J, R, r)] \\ &+\frac{\exp\,[-2JR^{1/2}(r-1)]}{2m^5R^{5/2}(1+2iK)(1+JR^{1/2})^2}[-RM^*(-m, J, R, r) \\ &+2(2K+S)N^*(-m, J, R, r)+SP^*(-m, J, R, r)] \\ &-\frac{\exp\,(mR^{1/2}r)}{2m^5R^{5/2}(1+2iK)(1+JR^{1/2})^2}A(-m, R, r)[RU(m, J, R) \\ &-2(2K+S)V(m, J, R)+SW(m, J, R)]\int_r^\infty \frac{\exp\,[-2J(r-1)-mr]R^{1/2}}{r}\,dr \\ &+\frac{\exp\,(-mR^{1/2}r)}{2m^5R^{5/2}(1+2iK)(1+JR^{1/2})^2}A(m, R, r)[RU(m, J, R) \\ &-2(2K+S)V(m, J, R)+SW(m, J, R)]\int_r^\infty \frac{\exp\,[-2J(r-1)+mr]R^{1/2}}{r}\,dr \end{aligned} \quad \ldots \quad (3.20)$$

where

$$A(m, R, r) = 1+\frac{3}{mR^{1/2}r}+\frac{3}{m^2Rr^2}$$

$$F^*(J, R, r) = \frac{5JR^{1/2}}{4r^2}+\frac{5}{2r^5}$$

$$G^*(J, R, r) = 5/4\,\frac{J^3R^{3/2}}{r^2}+\frac{5J^2R}{r^3}+\frac{15}{2}\,\frac{JR^{1/2}}{r^4}+\frac{15}{4r^5}$$

$$H^*(J, R, r) = \frac{5J^3R^{3/2}}{4r^2}+\frac{15}{2}\,\frac{J^2R}{r^3}+\frac{15JR^{1/2}}{r^4}+\frac{15}{2r^5}$$

$$M^*(m,\, J,\, R,\, r) = \frac{mR^{5/2}}{16r}(2J-m)+\frac{R^2}{16r^2}(8J^2m^2-4Jm^3+m^4)$$
$$+\frac{R^{3/2}}{8r^3}(12J^2m+6Jm^2-m^3)+\frac{3R}{8r^4}(4J^2+8Jm+m^2)$$
$$+\frac{3}{2}\frac{R^{1/2}}{r^5}(2J+m)+\frac{3}{2r^6}$$

$$N^*(m,\, J,\, R,\, r) = \frac{m^6R^{7/2}}{64r}(2J-m)+\frac{R^3}{64r^2}(32J^4m^2-16J^3m^3+8J^2m^4$$
$$-4Jm^5+m^6)+\frac{R^{5/2}}{32r^3}(48J^4m+56J^3m^2-20J^2m^3+6Jm^4-m^5)$$
$$+\frac{3R^2}{32r^4}(16J^4+64J^3m+36J^2m^2-8Jm^3+m^4)$$
$$+\frac{3R^{3/2}}{8r^5}(16J^3+32J^2m+10Jm^2-m^3)$$
$$+\frac{3R}{8r^6}(32J^2+36Jm+5m^2)+\frac{27R^{1/2}}{4r^7}(2J+m)+\frac{27}{4r^8}$$

$$P^*(m,\, J,\, R,\, r) = \frac{R^{7/2}}{32r}(-4J^3m^4+2J^2m^5+2Jm^6-m^7)$$
$$+\frac{R^3}{32r^2}(16J^4m^2-8J^3m^3+6J^2m^4-4Jm^5+m^6)$$
$$+\frac{R^{5/2}}{16r^3}(24J^4m+44J^3m^2-18J^2m^3+6Jm^4-m^5)$$
$$+\frac{3R^2}{16r^4}(8J^4+48J^3m+34J^2m^2-8Jm^3+m^4)$$
$$+\frac{3R^{3/2}}{4r^5}(12J^3+30J^2m+10Jm^2-m^3)$$
$$+\frac{3R}{4r^6}(30J^2+36Jm+5m^2)+\frac{27R^{1/2}}{2r^7}(2J+m)+\frac{27}{2r^8}$$

$$U(m,\, J,\, R) = \frac{m^6R^3(4J^2-m^2)}{16},$$

$$V(m,\, J,\, R) = \frac{m^4R^4(4J^2-m^2)}{64},$$

$$W(m,\, J,\, R) = \frac{R^4}{32}(8J^4m^4-6J^2m^6+m^8). \qquad \text{.. .. .. .. (3.21)}$$

D_1 and D_4 are determined from the boundary condition that

$$\chi_f = \frac{d\chi_f}{dr} = 0 \text{ on } r = 1. \qquad \text{.. .. .. .. (3.22)}$$

We note that in this case χ_f cannot be put in the form

$$\chi_f = \chi_f^{(1)}+(2K+S)\chi_f^{(2)}+S\chi_f^{(3)} \qquad \text{.. .. .. (3.23)}$$

where $\chi_f^{(1)}$, $\chi_f^{(2)}$ and $\chi_f^{(3)}$ do not contain K or S explicitly which is possible in the case of steady component of secondary flow. This is due to the presence of the term $\frac{1}{1+2iK}$, contributed by the acceleration gradient tensor, therefore we cannot linearly combine the effects of the Newtonian viscosity and the effects of the viscoelasticity and cross-viscosity to obtain the fluctuating component of the stream function in secondary flow.

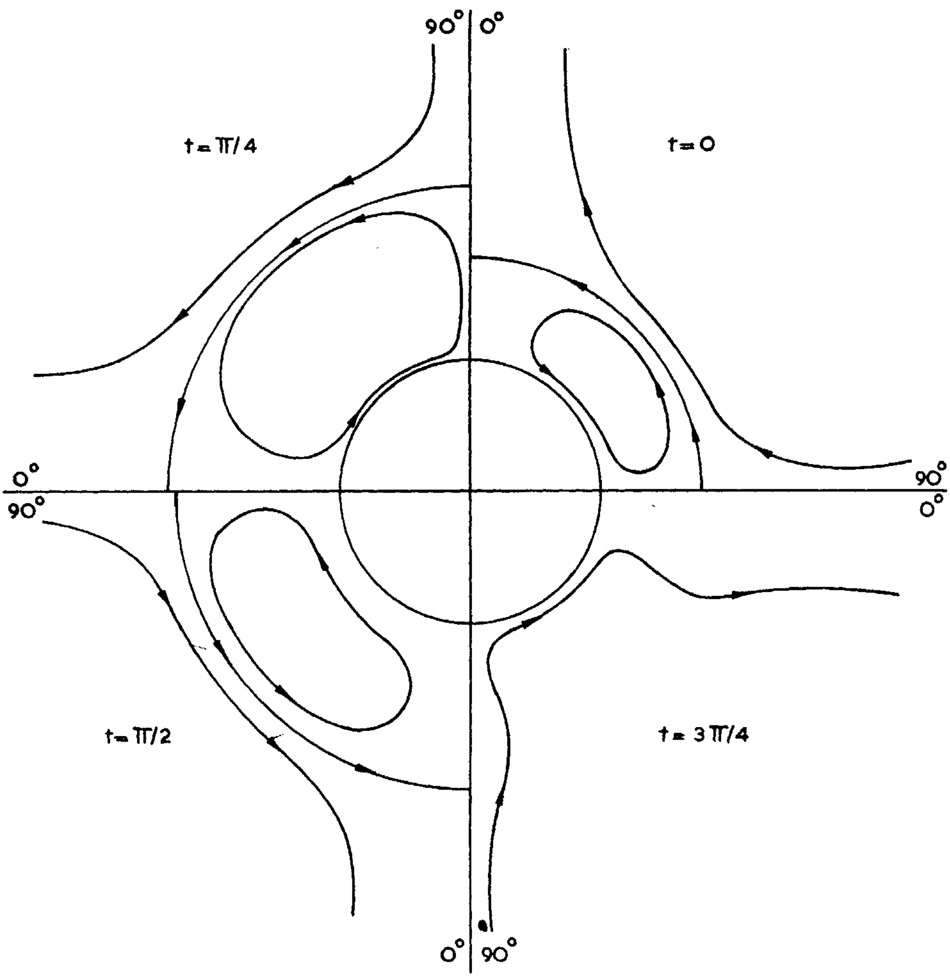

FIG. 1. $K = 0$, $S = 0$ and $R = 12$.

The stream function $\psi(r, \theta)$ given by

$$\psi(r, \theta) = \psi_s(r, \theta) + \psi_f(r, \theta)e^{2it}$$
$$= \sin^2\theta \cos\theta[\chi_s(r) + \chi_f(r)e^{2it}] \quad \ldots \quad \ldots \quad \ldots \quad (3.24)$$

specifies the stream function of the secondary motion for all time. We have studied in detail the flow field for three types of fluids

(i) Newtonian fluids
(ii) Viscoelastic fluids with $2K+S = 0$
(iii) Viscoelastic fluids with $2K+S \neq 0$.

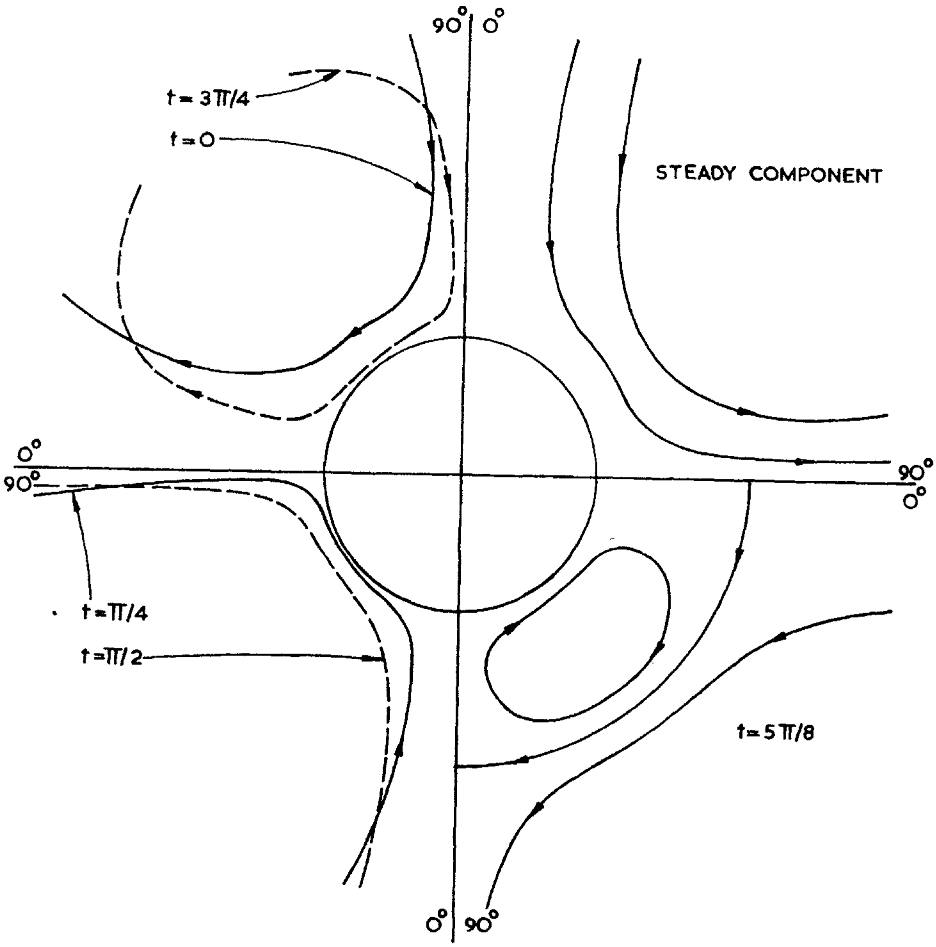

FIG. 2. $K = 0$, $S = 0$ and $R = 1$.

4. Discussion of Results

We have chosen for our numerical computation the three fluids whose fluid parameters are

(i) $K = 0, \quad S = 0$

(ii) $2K+S = 0, \quad \dfrac{K}{R} = \dfrac{\phi_2}{\rho a^2} = -0{\cdot}15, \quad \dfrac{S}{R} = \dfrac{\phi_3}{\rho a^2} = 0{\cdot}03$

(iii) $2K+S \neq 0, \quad \dfrac{K}{R} = -0{\cdot}01, \quad \dfrac{S}{R} = 0{\cdot}03$

so that $K < 0$ and $-2K < S < -4K$.

We have restricted our study up to Reynolds number 50 and up to a distance 3 from the centre of the sphere.

Discussing first the steady component, we notice that in this range of Reynolds numbers 1 to 50, all the three fluids exhibit a separation of the secondary flow field. The flow field in this case is separated into two regions by a streamline $r = r_c$. In the inner region, the fluid is drawn in at the pole

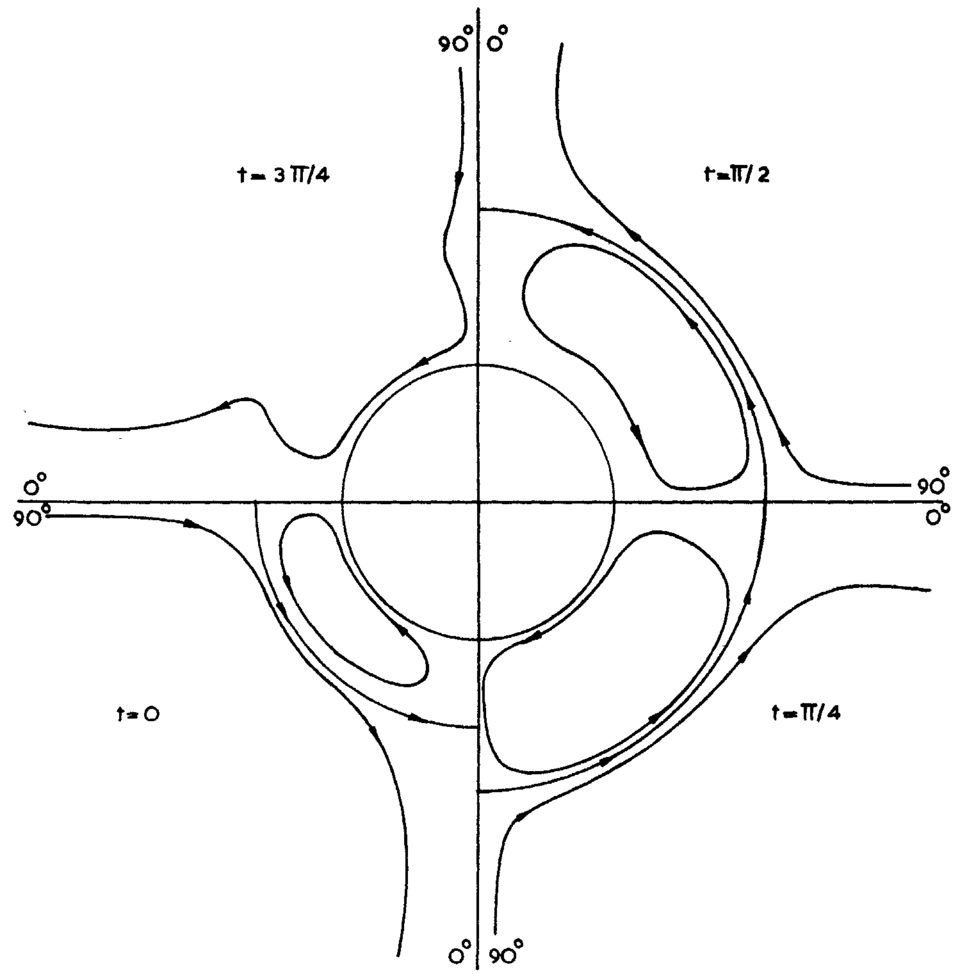

FIG. 3. $K = -0{\cdot}225$, $S = 0{\cdot}450$ and $R = 15$.

and thrown out at the equator, the streamlines forming closed loops. In the outer region, the curves are open, the fluid approaches the sphere at the equator and recedes at the pole. For the fluid (i) this separation is seen at $R = 12$ while for the fluid (ii) is evident only at $R = 15$ and for the fluid (iii) at $R = 10$. This separation is caused entirely by the motion of the boundary in the case of the Newtonian fluids. For types of the fluids (ii) an extra normal stress is introduced along streamlines in planes perpendicular to the axis of oscillation and this tends to delay the onset of separation. In fluids of the type (iii) two extra normal stresses are introduced, one along the streamlines in planes perpendicular to the axis of oscillation and the other in the radial direction. These two together help the onset of separation unlike the case

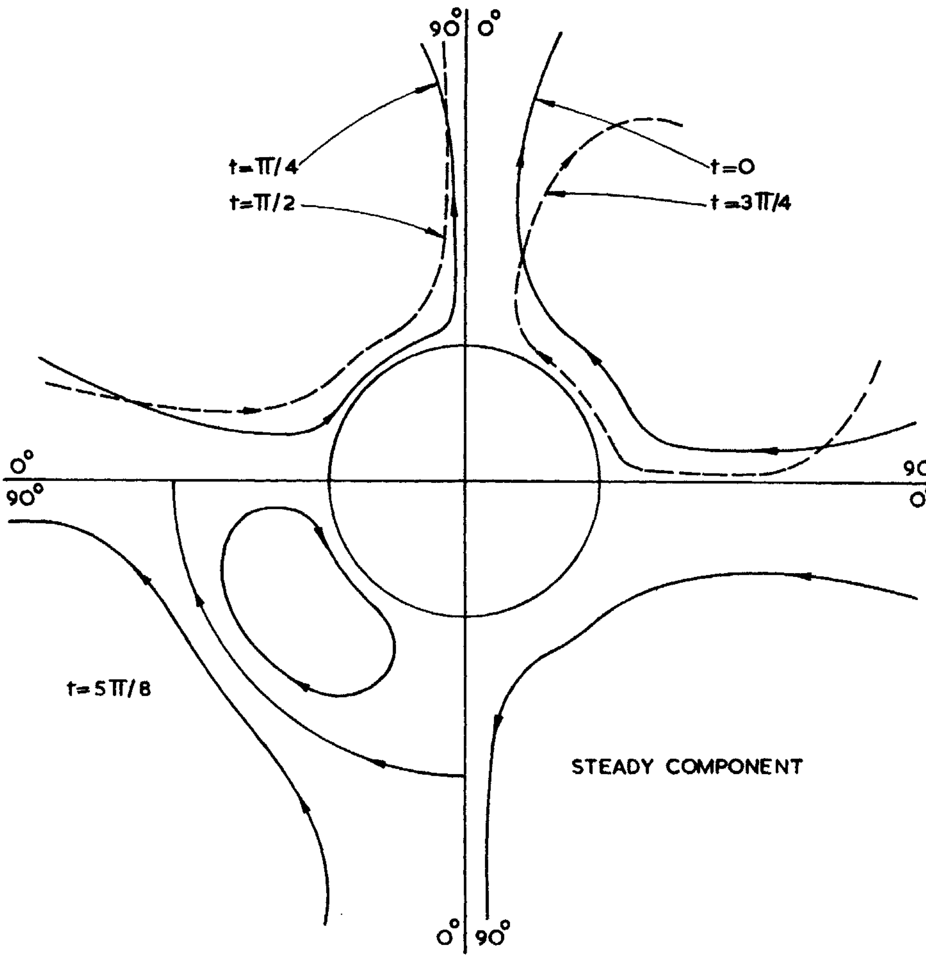

FIG. 4. $K = -0{\cdot}015$, $S = 0{\cdot}03$ and $R = 1$.

when only $T_{\phi\phi}$ is present. For Reynolds numbers less than these critical ones, the entire flow field is filled with the same type of streamlines. These curves are open and the fluid is drawn in at the pole and thrown away at the equator. For Reynolds numbers greater than the critical ones, the entire flow pattern is reversed in direction with open streamlines approaching the sphere at the equator and receding from it at the pole. This phenomenon is observed in all fluids caused primarily by the motion of the boundary and retarded or assisted as the case may be by the normal stress present in fluids of the types (ii) and (iii).

We have studied the unsteady component of secondary flow in the following cases:

Case 1: $R = 12$, $K = 0$, $S = 0$
Case 2: $R = 1$, $K = 0$, $S = 0$
Case 3: $R = 50$, $K = 0$, $S = 0$
Case 4: $R = 15$, $K = -0{\cdot}225$, $S = 0{\cdot}450$
Case 5: $R = 1$, $K = -0{\cdot}015$, $S = 0{\cdot}03$
Case 6: $R = 50$, $K = -0{\cdot}75$, $S = 1{\cdot}5$
Case 7: $R = 10$, $K = -0{\cdot}1$, $S = 0{\cdot}3$
Case 8: $R = 1$, $K = -0{\cdot}01$, $S = 0{\cdot}03$
Case 9: $R = 50$, $K = -0{\cdot}5$, $S = 1{\cdot}5$.

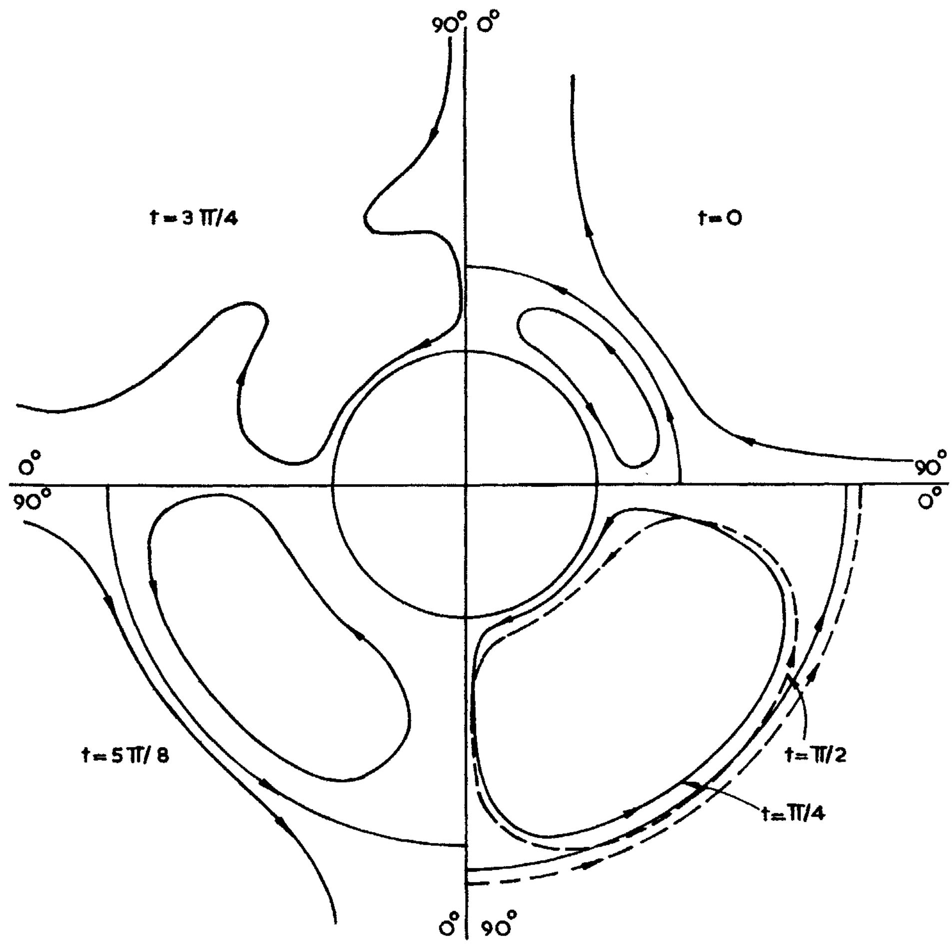

FIG. 5. $K = -0{\cdot}1$, $S = 0{\cdot}3$ and $R = 10$.

In all these cases t has been taken as $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}$ and $\frac{3\pi}{4}$.

We have noticed that the steady component of secondary flow is not representative of the entire secondary flow field. The steady component shows separation for the cases (1), (4) and (7) at $t = 0$. The separation is present in all the three cases in the secondary flow field but the circles of separation are much closer to the sphere than in the case of steady components. As t increases to $t = \frac{\pi}{4}$, the circle of separation recedes from the sphere till it recedes maximum distance and at $t = \frac{\pi}{2}$, it begins to approach the sphere again. At $t = \frac{3\pi}{4}$, the fluctuating component opposes and predominates over the steady one and the flow is no longer separated into regions with

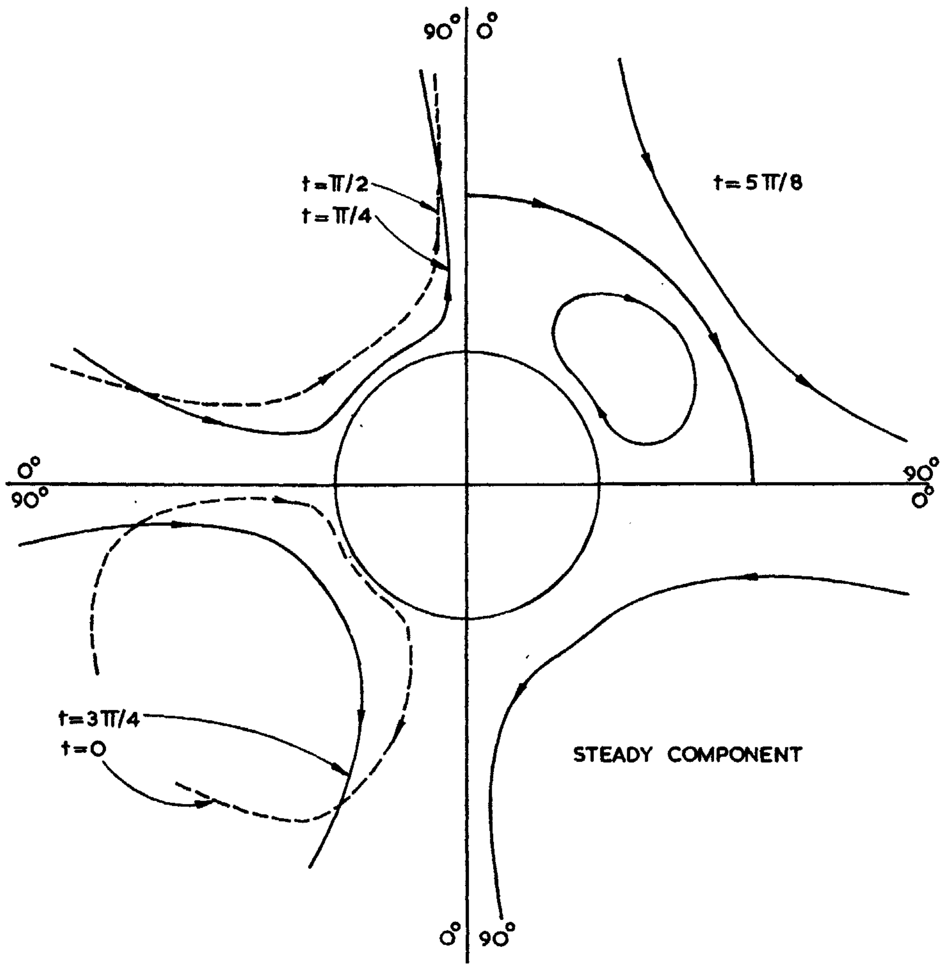

FIG. 6. $K = -0{\cdot}01$, $S = 0{\cdot}03$ and $R = 1$.

different flow directions. ψ is positive throughout and the streamlines are bent, close to the sphere because of the conflict between the steady and fluctuating components, ψ_s being negative and ψ_f being positive. As t increases the nature of the fluctuating component changes and separation sets in, close to the sphere. The entire cycle is repeated with period π. It is noticed that in all the three fluids, the steady component and fluctuating one are evenly matched and as time varies one or other predominates.

In cases (2), (5) and (8) for $R = 1$, for the three fluids the above statement does not hold. Here the steady component gives rise to open streamlines, the fluid being drawn in at the pole and thrown out at the equator. On considering the entire secondary flow field, we notice that at $t = 0$, the pattern is completely different. Open streamlines are present with reversed direction, the fluid coming in at the equator and thrown out at the poles. As t increases, separation sets in and at $t = \frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{2}$, close to the sphere, the streamlines approach the sphere at the poles and recede at the equator. At $t = \frac{5\pi}{8}$ separation sets in again and the streamlines pattern is reversed at $t = \frac{3\pi}{4}$. In this case, it is the fluctuating component which dominates the pattern of the secondary flow. As the sign of $\psi_f e^{2it}$ changes with t, the direction of the flow changes, the effect of ψ_s being hardly perceptible.

In cases (3), (6) and (9) for $R = 50$, fluctuating motion consists of open streamlines, the fluid being drawn in at the equator and thrown out at the poles. For all values of t, this pattern persists, when steady and fluctuating components are both taken into account. In these cases, the steady component has a much stronger effect than the fluctuating one. The fluctuating component has hardly any effect on the secondary flow field which is evident from the following table (case 3):

r	ψ_s	ψ $t = 0$	ψ $t = \pi/4$	ψ $t = \pi/2$	ψ $t = 3\pi/4$
1·2	2·53361	2·52775	2·44774	2·53947	2·61948
1·4	6·07404	6·09209	6·00570	6·05600	6·14238
1·6	8·83475	8·85071	8·78779	8·81880	8·88171
1·8	10·82304	10·83440	10·78664	10·81168	10·85944
2·0	12·26356	12·27253	12·23379	12·25459	12·29333

Similar results are observed for cases (6) and (9).

We notice that the fluctuating component predominates for small Reynolds numbers, is of equal importance as the steady component for a particular range of Reynolds number and has hardly any effect for larger Reynolds number. The secondary flow pattern for the three fluids considered shows no distinct difference, except that the separation sets in at different Reynolds numbers.

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