EIGENFUNCTIONS FOR THE SYMMETRICAL DEFORMATIONS OF A CYLINDER WHOSE PLANE FACES ARE DISPLACEMENT-FREE

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We present here the eigenfunction expansions of the bi-harmonic Love strain function for the axisymmetric deformations in a cylinder whose plane faces are displacement-free. The convergence for the simultaneous expansion of two arbitrary functions in terms of these functions is studied. As an example of these eigenfunctions, the symmetrical deformations in a cylinder whose plane faces are displacement-free and on whose curved edge radial displacement and transverse shear force are prescribed is worked out.

INTRODUCTION

Eigenfunction expansions of the bi-harmonic equation have been the subject of several interesting papers dealing with problems on rectilinear, circular and cylindrical geometries in the theory of elasticity. The convergence properties associated with these eigenfunction expansions have been discussed by Grinberg (1953), Nariboli (1965), Kolathaya and Ramachandra Rao (1969). It has been clearly brought out in these papers that formal expansions without the knowledge of the limiting values of the sums may lead to erroneous results. In view of this the eigenfunctions of the sector problem have been discussed by Ramachandra Rao (1969). We propose to initiate here a similar discussion for the axisymmetrical deformations in a cylinder whose plane faces are displacement-free.

It is well known that the eigenfunctions of the bi-harmonic equation do not possess any convenient orthogonality relation. While the eigenfunctions of the rectilinear and circular geometries admit a bi-orthogonality relation under certain boundary conditions, the non-availability of such a bi-orthogonality relation for the eigenfunctions of the cylindrical geometry limits the expansion procedure. In their paper on the 'end problem' of the cylinder Little and Childs (1967) overcome this difficulty by constructing an adjoint differential equation and adjusting the boundary conditions on this so as to obtain a bi-orthogonality relation between the eigenfunctions of the given problem and the adjoint problem. In this paper we first construct the eigenfunctions of the bi-harmonic equation for a cylinder whose plane faces are displacement-free. Secondly, following Little and Childs we construct the eigenfunctions of the adjoint problem and the orthogonality relation. This provides for the simultaneous expansion of two arbitrary functions. Finally, after discussing the convergence properties associated with these expansions, a closed-form solution for the symmetrical deformations in a cylinder when arbitrary radial displacement and transverse shear force are prescribed is worked out.

Basic Equations

We shall assume that the cylinder occupies the region $r \leq a$, $|z| \leq 1$ and that the plane faces $z = \pm 1$ are displacement-free. Along the curved surface of the cylinder $r = a$, we shall assume that the radial displacement $u(z)$ and transverse shear force $\tau(z)$ are prescribed. The governing differential equation and the boundary conditions in terms of the Love's strain function χ can be written as follows:

$$\nabla^4 \chi = 0, \quad r \leq a, \quad |z| \leq 1 \qquad \qquad (1)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

$$\left.\begin{aligned} U_r &= -\frac{1+\sigma}{E}\frac{\partial^2 \chi}{\partial r\, \partial z} = 0 \\ U_z &= \frac{1+\sigma}{E}\left\{2(1-\sigma)\nabla^2\chi - \frac{\partial^2 \chi}{\partial z^2}\right\} = 0 \quad \text{on } z = \pm 1 \end{aligned}\right\} \qquad (2)$$

and

$$\left.\begin{aligned} U_r &= -\frac{1+\sigma}{E}\frac{\partial^2 \chi}{\partial r\, \partial z} = u(z) \\ t_{rz} &= \frac{\partial}{\partial r}\left\{(1-\sigma)\nabla^2\chi - \frac{\partial^2 \chi}{\partial z^2}\right\} = \tau(z) \quad \text{on } r = a \end{aligned}\right\} \qquad (3)$$

The constant σ and E denote the Poisson's ratio and the Young's modulus respectively.

We assume that the solution of (1) is given by

$$\chi = \sum A_n \frac{I_0(\lambda_n r)}{\lambda_n I_1(\lambda_n a)} f_n(z) \qquad \qquad (4)$$

where I_0, I_1 are modified Bessel functions of the first kind and the A_n's are a set of complex constants. The f_n's which are the solutions of the differential equation

$$f^{\text{iv}} + 2\lambda^2 f'' + \lambda^4 f = 0 \qquad \qquad (5)$$

are subject to the boundary conditions

$$\left.\begin{aligned} f' &= 0 \\ f'' + \left(\frac{1+k}{1-k}\right)\lambda^2 f &= 0 \quad \text{on } z = \pm 1 \end{aligned}\right\} \qquad (6)$$

where

$$k = \frac{1}{3-4\sigma}.$$

The 'even solutions' for f_n in the case of symmetrical deformations are given by

$$f_n = (\sin \lambda_n + \lambda_n \cos \lambda_n) \cos \lambda_n z + \lambda_n z \sin \lambda_n \sin \lambda_n z \quad \ldots \quad \ldots \quad (7)$$

where the λ_n's are the roots of the transcendental equation

$$\phi(\lambda) = \sin 2\lambda + 2k\,\lambda = 0. \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

It may be noted that the summation in (4) is extended over the eigenvalues of (8) whose real parts are positive. The eqn. (8) has an infinite number of complex roots symmetrically located over the four quadrants in the complex λ-plane. The asymptotic form for the roots of (8) is given by

$$\lambda_n \simeq \frac{\alpha_n}{2} - \frac{\ln (2k\,\alpha_n)}{2\alpha_n} + \frac{i}{2} \ln (2k\alpha_n),$$

where

$$\alpha_n = \frac{(4n-1)\pi}{2}.$$

The roots of (8) can be calculated to any desired degree of accuracy using the asymptotic form for λ_n as the starting point and then followed by Newton's iterative method.

It may be noted that $\lambda = 0$ is an admissible eigenvalue. The corresponding eigenfunction is taken as unity.

Orthogonality Relation

The functions f_n given by (7) do not satisfy any orthogonal property. For this we construct the adjoint problem in such a way that the functions f_n and the eigenfunctions g_n of the adjoint differential equation together form a bi-orthogonal relation. We shall assume that the eigenfunctions g_n satisfy the differential equation

$$g^{\text{iv}} + 2\mu^2 g'' + \mu^4 g = 0.$$

Let g_m be the eigenfunction corresponding to the eigenvalue μ_m. Then we have

$$f_n^{\text{iv}} + 2\lambda_n^2 f_n'' + \lambda_n^4 f_n = 0$$

$$g_m^{\text{iv}} + 2\mu_m^2 g_m'' + \mu_m^4 g_m = 0.$$

Multiplying the first of the above equations by g_m and the second by f_n, subtracting and integrating between -1 and $+1$, we get from the boundary condtions (6)

$$\mu_m^2 f_n''' g_m - \lambda_n^2 f_n \left\{ g_m''' - \frac{3k-1}{1-k} g_m' \right\} + \int_{-1}^{+1} (f_n'' g_m'' - \lambda_n^2 \mu_m^2 f_n g_m)\, dz = 0.$$

The terms obtained by integrating by parts can be made to vanish by imposing the following boundary conditions on the functions g_m:

$$g_m = 0$$

$$g_m''' - \frac{3k-1}{1-k}\lambda_m^2 g_m' = 0.$$

Thus the bi-orthogonality between the functions f_n and g_m is given by

$$\int_{-1}^{+1} (f_n'' g_m'' - \lambda_n^2 \mu_m^2 f_n g_m)\, dz = 0. \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

The 'even solutions' for g_m satisfying the above boundary conditions are given by

$$g_m = \sin \mu_m \cos \mu_m z - z \cos \mu_m \sin \mu_m z \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where the μ_m's also satisfy the same transcendental eqn. (8). This is in conformity with the result that the given problem and the adjoint problem have the same characteristic equation. Therefore, we shall use the letter λ instead of μ throughout.

Eigenfunction Expansions and Convergence

We shall now consider the simultaneous expansion of two arbitrary functions $F(z)$ and $G(z)$ in terms of the functions f_n in the form

$$\left. \begin{aligned} F &= \Sigma A_n f_n' \\ G &= \Sigma A_n \lambda_n^2 f_n \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where the A_n's are an infinite set of complex constants to be determined using the bi-orthogonal relation (9). From (9) and (11) it follows that

$$A_n = \frac{\int_{-1}^{+1} (F' g_n'' - \lambda_n^2 G g_n)\, dz}{K_n} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where

$$K_n = \int_{-1}^{+1} (f_n'' g_n'' - \lambda_n^4 f_n g_n)\, dz.$$

Upon simplification it can be shown that

$$K_n = \lambda_n^3 \cos^2 \lambda_n \phi_n' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where ϕ_n' denotes the derivative of the characteristic function $\phi(\lambda_n) = \sin 2\lambda_n + 2k\lambda_n$. We have expressed K_n to contain the derivative of the characteristic function as one of the factors. Using the definition of A_n we can now easily sum the complex summations under (11). Using (10), (12) and (13) the constants A_n can be written as

$$A_n = \int_{-1}^{+1} \left\{ -\frac{(F'+G) g_n(\zeta)}{\lambda_n \cos^2 \lambda_n \phi_n'} - \frac{2F' \cos \lambda_n \zeta}{\lambda_n^2 \cos \lambda_n \phi_n'} \right\} d\zeta. \quad \ldots \quad \ldots \quad (14)$$

Using (14) as the definition of A_n, we shall now consider the convergence of the expansion procedure (11) by finding the limiting values of the sums of the series

$$s_n(z) = \sum_1^n \int_{-1}^{+1} \left[\frac{\{F'(\zeta)+G(\zeta)\}g_n(\zeta)h_n(z)}{\cos^2 \lambda_n\phi_n'} + \frac{2F'(\zeta)\cos\lambda_n\zeta h_n(z)}{\lambda_n \cos\lambda_n\phi_n'}\right] d\zeta$$

and

$$t_n(z) = \sum_1^n \int_{-1}^{+1} \left[-\frac{\{F'(\zeta)+G(\zeta)\}\lambda_n g_n(\zeta) f_n(z)}{\cos^2 \lambda_n\phi_n'} - \frac{2F'(\zeta)\cos\lambda_n\zeta f_n(z)}{\cos\lambda_n\phi_n'}\right] d\zeta \tag{15}$$

where

$$h_n(z) = (\cos\lambda_n + \lambda_n k \sin\lambda_n)\sin\lambda_n z + k\lambda_n z\cos\lambda_n\cos\lambda_n z.$$

Interchanging the order of summation and integration we can write the above sums as

$$s_n(z) = \int_{-1}^{+1} \left[\{F'(\zeta)+G(\zeta)\}S_n(z,\zeta) + 2F'(\zeta)T_n(z,\zeta)\right] d\zeta$$

$$t_n(z) = -\int_{-1}^{+1} \left[\{F'(\zeta)+G(\zeta)\}U_n(z,\zeta) + 2F'(\zeta)V_n(z,\zeta)\right] d\zeta$$

where

$$S_n = \sum \frac{g_n(\zeta)h_n(z)}{\cos^2\lambda_n\phi_n'}, \quad T_n = \sum \frac{\cos\lambda_n\zeta h_n(z)}{\lambda_n\cos\lambda_n\phi_n'}$$

$$U_n = \sum \frac{\lambda_n g_n(\zeta) f_n(z)}{\cos^2\lambda_n\phi_n'}, \quad V_n = \sum \frac{\cos\lambda_n\zeta f_n(z)}{\cos\lambda_n\phi_n'}.$$

To evaluate the above sums we shall consider the following contour integrals, viz.

$$J_1 = \frac{1}{2\pi i}\int_c \frac{g(\lambda,\zeta)h(\lambda,z)}{\cos^2\lambda\phi(\lambda)}; \quad J_2 = \frac{1}{2\pi i}\int_c \frac{\cos\lambda\zeta h(\lambda,z)}{\lambda\cos\lambda\phi(\lambda)}\,d\lambda$$

$$I_1 = \frac{1}{2\pi i}\int_c \frac{\lambda g(\lambda,\zeta)f(\lambda,z)}{\cos^2\lambda\phi(\lambda)}\,d\lambda; \quad I_2 = \frac{1}{2\pi i}\int_c \frac{\cos\lambda\zeta f(\lambda,z)}{\cos\lambda\phi(\lambda)}\,d\lambda$$

where c is a large semicircle to the right of the imaginary axis with centre at the origin and

$$f(\lambda, z) = \{(1-k)\cos\lambda + \lambda k\sin\lambda\}\cos\lambda z - \lambda\, kz\cos\lambda\sin\lambda z$$

$$g(\lambda,\zeta) = \sin\lambda\cos\lambda\zeta - \cos\lambda\sin\lambda\zeta$$

$$h(\lambda, z) = (\cos\lambda + \lambda k\sin\lambda)\sin\lambda z + \lambda k\, z\cos\lambda\cos\lambda z.$$

The integrals along the imaginary axis cancel out as the integrands are odd in λ. Also the integrals along the semicircle tend to zero in the limit. The poles of the integrand are given by the zeros of $\phi(\lambda) = 0$, i.e. the roots of the characteristic equation, $\cos\lambda = 0$, i.e. $\lambda = \frac{n\pi}{2}$ where n is an odd integer and the poles at the origin. Evaluating the residues at the simple and double poles, we get

$$S_n = 0$$

$$T_n = -\frac{z}{4}+\frac{1}{2}\sum_{n=1,3,5}\frac{\sin\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}}{(n\pi/2)}$$

$$U_n = -\sum_{n=1,3,5}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}$$

$$V_n = -\frac{1-k}{4(1+k)}+\frac{1}{2}\sum_{n=1,3,5}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}.$$

Substituting the values of the above sums in (15) we get

$$\lim_{n\to\infty} s_n(z) = \int_{-1}^{+1} F'(\zeta)\left\{-\frac{z}{2}+\sum_{1}^{\infty}\frac{\sin\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}}{n\pi/2}\right\}d\zeta \quad \ldots \quad \ldots \quad (16)$$

$$\lim_{n\to\infty} t_n(z) = \int_{-1}^{+1}\left[\{F'(\zeta)+G(\zeta)\}\sum_{1}^{\infty}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}\right]d\zeta$$

$$+\int_{-1}^{+1}F'(\zeta)\left[\frac{1-k}{2(1+k)}-\sum_{1}^{\infty}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}\right]d\zeta. \quad (17)$$

Upon integrating under the summation sign, it follows from (16)

$$\lim_{n\to\infty} s_n(z) = -\frac{z}{2}\{F(1)-F(-1)\}+\int_{-1}^{+1}F(\zeta)\sum_{n=1}^{\infty}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}\,d\zeta.$$

Relation (17) can be rewritten as

$$\lim_{n\to\infty} t_n(z) = \frac{1-k}{2(1+k)}\{F(1)-F(-1)\}+\int_{-1}^{+1}G(\zeta)\sum_{n=1}^{\infty}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}\,d\zeta$$

$F(1) = F(-1) = 0$ may be taken as consistency conditions, a feature common to the Fourier series. The limiting value of the sums $s_n(z)$ and $t_n(z)$ can, therefore, be written as

$$\lim_{n\to\infty} s_n(z) = \int_{-1}^{+1}F(\zeta)\sum_{n=1}^{\infty}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}\,d\zeta$$

$$\lim_{n\to\infty} t_n(z) = \int_{-1}^{+1}G(\zeta)\sum_{n=1}^{\infty}\cos\frac{n\pi z}{2}\cos\frac{n\pi\zeta}{2}\,d\zeta.$$

It, therefore, follows that if the Fourier series corresponding to $F(z)$ and $G(z)$ converge to the value of the functions

$$\lim_{n\to\infty} s_n(z) = F(z)$$

and

$$\lim_{n \to \infty} t_n(z) = G(z),$$

thus establishing the convergence of the expansions (11).

The above expansion will give a closed form solution for the axisymmetric deformations in a cylinder when radial displacement $u(z)$ and shear stress $\tau(z)$ are prescribed on the edge $r = a$. Identifying the functions $F(z)$ and $G(z)$ with the prescribed functions $u(z)$ and $\tau(z)$ we have

$$F = -\frac{E}{1+\sigma} u \quad \text{and} \quad G = \frac{1}{1-\sigma} - \frac{\sigma E}{1-\sigma^2} u'. \qquad \ldots \quad \ldots \quad (18)$$

The solution of the problem is now given by eqns. (4), (12) and (18).

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