UNSTEADY LAMINAR MOTION OF AN ELECTROCONDUCTING LIQUID UNDER THE ACTION OF A TRANSVERSE MAGNETIC FIELD

by K. N. SHAH, *Bhavan's R.A. College of Science, Ahmedabad*

(Communicated by F. C. Auluck, F.N.A.)

(*Received 20 January 1970*)

Steady Laminar Motion of an electroconducting liquid between two parallel planes under a transverse magnetic field has been discussed by Cowling, Ferraro and others. In this paper, an attempt is made to discuss the unsteady motion. When the pressure gradient together with external force in the direction of motion is constant, the motion is steady but if it is time-varying the motion is unsteady. Method of Laplace's transform has been applied. Special cases are discussed when the time-varying pressure gradient together with external force is constant and when it is exponential in time.

1. INTRODUCTION

Ferraro, Cowling and Plumpton (1966), Cowling (1957) and Hartmann and Lazarus (1938) have discussed the steady motion of an electroconducting liquid between two parallel plates under the action of a transverse magnetic field.

In this paper an attempt is made to study the unsteady motion between two parallel plates when the pressure gradient together with the external force is a function of time. The method of Laplace's transform is applied. The expressions for the velocity and the magnetic component in its direction are obtained in this general case. The motion is taken as steady in the initial state.

Special cases are discussed when the time-varying pressure gradient together with the external force is an exponentially increasing as well as an exponentially decreasing function of time. The steady solution of Ferraro, Cowling and others is also deduced as a special case.

2. FUNDAMENTAL EQUATIONS

If $\bar{v}$ denote the velocity of an electroconducting liquid, ρ the density, γ coefficient of kinematical viscosity, p the pressure at a particular point, $\bar{F}$ the sum of the external forces, the term $\bar{J}\times\bar{B}$ the mechanical force exerted by the magnetic field $\bar{B}$ on the volume-element of the conductor carrying a current J, σ the coefficient of conductivity and μ the coefficient of permeability, then the equations satisfied by these quantities are

$$\text{curl } \bar{H} = \bar{J} \quad \ldots \quad (1)$$

$$\text{curl } \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \quad \bar{B} = \mu \bar{H} \quad \ldots \quad (2)$$

$$\bar{J} = \sigma(\bar{E} + \mu \bar{v} \times \bar{H}) \quad \ldots \quad (3)$$

$$\rho \cdot \frac{\partial \bar{v}}{\partial t} + \rho(\bar{v} \,.\, \overline{\nabla})\bar{v} = \rho \bar{F} - \nabla p + \rho \gamma \nabla^2 \bar{v} + \bar{J} \times \bar{B} \quad \ldots \quad (4)$$

$$\text{div } \bar{v} = 0; \quad \text{div } \bar{B} = 0. \quad \ldots \quad (5)$$

Equations (2) and (3) reduce to

$$\frac{\partial \bar{B}}{\partial t} = \text{curl } (\bar{v} \times \bar{B}) + \frac{1}{\mu\sigma} \nabla^2 B. \quad \ldots \quad (6)$$

3. Equations for Laminar Motion

Let us apply these equations to the motion between two parallel plates. The liquid is constrained to move horizontally between two infinite horizontal stationary walls. Take rectangular axes $Oxyz$ such that the bounding walls have equation $z = \pm L$. Assume the fluid to have a velocity $[v(z, t), 0, 0]$. The seed magnetic field is given by $\bar{B}_0 = (0, 0, B_0)$, $\bar{B}_0 = \mu \bar{H}_0$. The equation of continuity div $\bar{v} = 0$ is satisfied.

Because the liquid near the walls moves at a slower rate than the liquid near the plane $z = 0$, it tends to pull out the lines of force in the direction of motion, the component $\bar{b} = [b(z, t), 0, 0]$ is created, $\bar{b} = \mu \bar{h}$. The resulting magnetic force is $\bar{B} = \bar{B}_0 + \bar{b}$. Equations (1) to (6) will now give

$$0 = J_x = 0, \ \frac{1}{\mu} \frac{\partial b}{\partial Z} = J_y = \sigma[E_y - B_0 \bar{v}], \ 0 = J_z = \frac{\partial b}{\partial y} \quad \ldots \quad (7)$$

$$\rho \frac{\partial v}{\partial t} = \rho F_x - \frac{\partial p}{\partial x} + \rho \gamma \frac{\partial^2 v}{\partial Z^2} + \frac{B_0}{\mu} \frac{\partial b}{\partial Z} \quad \ldots \quad (8)$$

$$\mu\sigma \frac{\partial b}{\partial t} = \mu\sigma B_0 \frac{\partial v}{\partial Z} + \frac{\partial^2 b}{\partial Z^2} \quad \ldots \quad (9)$$

$$0 = \rho F_y - \frac{\partial p}{\partial y} = \rho F_z - \frac{\partial p}{\partial Z} - \frac{b}{\mu} \frac{\partial b}{\partial Z}. \quad \ldots \quad (10)$$

If

$$\rho F_x - \frac{\partial p}{\partial_x} = \rho f'(t) \text{ and } F_y = 0$$

the external force $\rho \bar{F}$ must be a gradient of a scalar ϕ. Equation (10) integrates to

$$p = p_0(x, t) - \frac{b^2}{2\mu} + \phi + \text{constant} \quad \ldots \quad (10a)$$

$$\frac{\partial p}{\partial x} = \frac{\partial p_0}{\partial x} + \frac{\partial \phi}{\partial x}; \quad p_0 = -\rho f'(t) \,.\, x.$$

The pressure equation is

$$p = -\rho f'(t) + p_1(\tau) + \phi \qquad \text{.. .. .. (10b)}$$

where

$$p_1(\tau) = \text{constant} - \frac{b^2}{2\mu}.$$

Equation (8) reduces to

$$\rho \frac{\partial v}{\partial t} = \rho f'(t) + \rho\gamma \frac{\partial^2 v}{\partial Z^2} + \frac{B_0}{\mu} \frac{\partial b}{\partial Z}. \qquad \text{.. .. .. (11)}$$

Let

$$V = v - f(t) + f(0). \qquad \text{.. .. .. .. (12)}$$

Then eqn. (11) reduces to

$$\rho \frac{\partial V}{\partial t} = \rho\gamma \frac{\partial^2 V}{\partial Z^2} + \frac{B_0}{\mu} \frac{\partial b}{\partial Z} \qquad \text{.. .. .. .. (13)}$$

and eqn. (9) can be written as

$$\mu\sigma \frac{\partial b}{\partial t} = \mu\sigma B_0 \frac{\partial V}{\partial Z} + \frac{\partial^2 b}{\partial Z^2}. \qquad \text{.. .. .. (14)}$$

(13) and (14) are the two equations satisfied by V and b.

4. Solutions of (13) and (14)

We use the method of Laplace's transform to solve eqns. (13) and (14).

Assume that V and b are of exponential orders η_1 and η_2 respectively. Let $\eta_0 = \max. (\eta_1, \eta_2)$

Laplace's transforms

$$\bar{V} = \int_0^\infty V \,.\, e^{-st}\, dt \quad \text{and} \quad \bar{b} = \int_0^\infty b \,.\, e^{-st}\, dt$$

exists if $s > \eta_0$ (s is taken to be real).

Multiply both the sides of eqns. (13) and (14) by e^{-st} and integrate with respect to t from '0' to '∞'. Taking the initial values of V and b as V_0 and b_0 respectively, we get

$$-V_0 + s\bar{V} = \gamma \frac{\partial^2 \bar{V}}{\partial Z^2} + \frac{B_0}{\mu\rho} \frac{\partial \bar{b}}{\partial Z} \qquad \text{.. .. .. (15)}$$

$$-\mu\sigma b_0 + \mu\sigma s\bar{b} = \mu\sigma B_0 \frac{\partial \bar{V}}{\partial Z} + \frac{\partial^2 b}{\partial Z^2}. \qquad \text{.. .. .. (16)}$$

In usual notations, they reduce to

$$\frac{B_0}{\mu\rho} D\bar{b} + (\gamma D^2 - s)\bar{V} = -V_0 \qquad \text{.. .. .. .. (17)}$$

$$(D^2 - \mu\sigma s)\bar{b} + \mu B_0 D\bar{V} = -\mu\sigma b_0 \qquad \text{.. .. .. (18)}$$

where

$$D \equiv \frac{d}{dZ}.$$

Assuming the motion to be steady initially and that the magnetic component initially is such that $\frac{\partial b}{\partial t} = 0$, we get the initial values of V and b as V_0 and b_0 given by

$$V_0 = r\left[1-\frac{\cosh\frac{MZ}{L}}{\cosh M}\right]-f(0), \quad b_0 = r\,.\frac{\gamma M}{L}\,.\tanh M\,.\frac{\mu\bar{\rho}}{B_0}\left[\frac{Z}{L}-\frac{\sinh\frac{MZ}{L}}{\sinh M}\right] \qquad (19)$$

$$r = \frac{f'(t)L}{\gamma\,.\frac{M}{L}\,.\tanh M}.$$

Eliminating $\bar{b}$ between (17) and (18), we get

$$\left[(\gamma D^2-s)(D^2-\mu\sigma s)-\frac{\sigma B_0^2}{\rho}D^2\right]\bar{V} = r\left[(1+\mu\sigma\gamma)\frac{M^2}{L^2}-\mu\sigma s\right]\frac{\cosh\frac{MZ}{L}}{\cosh M}$$
$$+\mu\sigma s r-\mu\sigma s f(0)-\mu\sigma f'(0). \qquad .. \quad (20)$$

If α^2 and β^2 are the roots of

$$\gamma x^2-\left\{(\mu\sigma\gamma+1)s+\frac{\sigma B_0^2}{\rho}\right\}x+\mu\sigma s^2 = 0 \qquad .. \qquad .. \qquad .. \quad (21)$$

and A, B, C, D be constant to be determined by the boundary conditions, the solution of (20) is

$$\bar{V} = A\cosh\alpha Z+B\sinh\alpha Z+C\cosh\beta Z+D\sinh\beta Z+\frac{V_0}{s}-\frac{f'(0)}{s^2}. \qquad (22)$$

Putting this value of $\bar{V}$ in (17), we get

$$\frac{B_0}{\mu\rho}D\bar{b} = A(s-\gamma\alpha^2)\cosh\alpha Z+B(s-\gamma\alpha^2)\sinh\alpha Z+C(s-\gamma\beta^2)\cosh\beta Z$$
$$+D(s-\gamma\beta^2)\sinh\beta Z+\gamma\,.\frac{M^2}{L^2}\,.\frac{\cosh\frac{MZ}{L}}{\cosh M}\frac{r}{s}-\frac{f'(0)}{s}. \qquad .. \qquad .. \quad (23)$$

Integrating with respect to Z, using

$$\frac{s-\gamma\alpha^2}{\alpha} = K_\alpha, \quad \frac{s-\gamma\beta^2}{\beta} = K_\beta$$

we obtain

$$\left.\begin{aligned} \frac{B_0}{\mu\rho}\bar{b} &= AK_\alpha\sinh\alpha Z+BK_\alpha\cosh\alpha Z+CK_\beta\sinh\beta Z+DK_\beta\cosh\beta Z \\ &\quad +\frac{\gamma M}{L}\,.\frac{r}{s}\tanh M.\frac{\sinh\frac{MZ}{L}}{\sinh M}-\frac{f'(0)}{s}\,.Z \\ &= AK_\alpha\sinh\alpha Z+BK_\alpha\cosh\alpha Z+CK_\beta\sinh\beta Z+DK_\beta\cosh\beta Z+\frac{b_0}{s}. \end{aligned}\right\} \qquad (24)$$

Applying the boundary conditions

$$\bar{V} = -\int_0^\infty e^{-st}.f(t)\,dt + \frac{f(0)}{s} = -F(s) + \frac{f(0)}{s} \text{ and } \bar{b} = 0 \text{ on } Z = \pm L \quad (25)$$

where

$$F(s) = \int_0^\infty e^{-st}.f(t)\,dt,$$

we get

$$A \cosh \alpha L \pm B \sinh \alpha L + C \cosh \beta L \pm D \sinh \beta L = -F(s) + \frac{f'(0)}{L^2} \quad .. \quad (62)$$

$$\pm AK_\alpha \sinh \alpha L + BK_\alpha \cosh \alpha L \pm CK_\beta \cosh \beta L + D \sinh \beta L = 0 \quad .. \quad (27)$$

$$A = \frac{\left(F(s) - \frac{f'(0)}{s^2}\right) K_\beta \sinh \beta L}{K_\alpha \sinh \alpha L \cosh \beta L - K_\beta \cosh \alpha L \sinh \beta L}, \quad C = \frac{-\left(F(s) - \frac{f'(0)}{s^2}\right) K_\alpha \sinh \alpha L}{K_\alpha \sinh \alpha L \cosh \beta L - K_\beta \cosh \alpha L \sinh \beta L} \quad .. \quad (28)$$

$$B = 0 = D. \quad .. \quad .. \quad .. \quad .. \quad (29)$$

Putting these values of A, B, C, D in (23) and (24), we obtain

$$\bar{V} = \frac{V_0}{s} - \left[F(s) - \frac{f'(0)}{s^2}\right] \frac{K_\alpha \sinh \alpha L \cosh \beta Z - K_\beta \cosh \alpha Z \sinh \beta L}{K_\alpha \sinh \alpha L \cosh \beta L - K_\beta \cosh \alpha L \sinh \beta L} \quad .. \quad (30)$$

and

$$\bar{b} = \frac{\mu\rho}{B_0}\left[\frac{b_0}{s} + \left(F(s) - \frac{f'(0)}{s^2}\right) K_\alpha K_\beta \frac{\sinh \alpha Z \sinh \beta L - \sinh \beta Z \sinh \alpha L}{K_\alpha \sinh \alpha L \cosh \beta L - K_\beta \cosh \alpha L \sinh \beta L}\right]. \quad (31)$$

We now find the inverse of Laplace's transforms $\bar{V}$ and $\bar{b}$ as V and b given by

$$V = L^{-1}\bar{V} \text{ and } b = L^{-1}\bar{b}.$$

We get

$$V = V_0 - L^{-1}\left(F(s) - \frac{f'(0)}{s^2}\right) \frac{K_\alpha \sinh \alpha L \cosh \beta Z - K_\beta \cosh \alpha Z \sinh \beta L}{K_\alpha \sinh \alpha L \cosh \beta L - K_\beta \cosh \alpha L \sinh \beta L} \quad .. \quad (32)$$

$$b = \frac{\mu\rho}{B_0}\left[b_0 + L^{-1}\left(F(s) - \frac{f'(0)}{s^2}\right) K_\alpha K_\beta \frac{\sinh \alpha Z \sinh \beta L - \sinh \beta Z \sinh \alpha L}{K_\alpha \sinh \alpha L \cosh \beta L - K_\beta \cosh \alpha L \sinh \beta L}\right]. \quad (33)$$

5. Special Cases

Case (i): $\rho f'(t) = \text{constant } \rho E = f'(0) \begin{bmatrix} f(t) = Et \\ F(s) = E/s^2 \end{bmatrix}$

$$V = v - Et$$

$$V = V_0$$

$$\therefore \quad v = v_0$$

$$b = \frac{\mu\rho}{B_0}.$$

This result proves that, when the pressure gradient together with the external force is constant, the motion is steady.

The time-varying pressure gradient together with the external force is responsible for creating the unsteadiness.

Case (ii): $f(t) = e^{mt}$ (when m is positive and when m is negative).

The following discussion of the nature of the roots of eqn. (21) is useful.

TABLE I

Sl. No.	Range of s	Nature of roots of eqn. (21)
1	$s > 0$ and $s < 0$, $s > \dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}+1)^2}$	Roots are real, positive, unequal
2	$\dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}-1)^2} < s < \dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}+1)^2}$	Roots are complex
3	$s < \dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}-1)^2}$	Roots are real, negative, unequal
4	$s = \dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}-1)^2}$	Roots are real, equal, positive
5	$s = \dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}+1)^2}$	Roots are real, equal, negative

We can find the values of V and b for all the five cases in Table I. Only two cases, (1) and (3),

(i) $f(t) = e^{mt}$, m is positive

(ii) $F(s) = \dfrac{1}{s-m}$ when $s > m$, $F(s)$ does not exist when $s \leqslant m$.

$$V = V_0 - f'(0)\,t - e^{mt}\cdot\frac{K_{\alpha_m}\sinh\alpha_m L\cosh\beta_m Z - K_{\beta_m}\sinh\beta_m L\cosh\alpha_m Z}{K_{\alpha_m}\sinh\alpha_m L\cosh\beta_m L - K_{\beta_m}\sinh\beta_m L\cosh\alpha_m L} \quad \ldots (34)$$

$$v = v_0 - f'(0)\,t + e^{mt}\left[1 - \frac{K_{\alpha_m}\sinh\alpha_m L\cosh\beta_m Z - K_{\beta_m}\sinh\beta_m L\cosh\alpha_m Z}{K_{\alpha_m}\sinh\alpha_m L\cosh\beta_m L - K_{\beta_m}\sinh\beta_m L\cosh\alpha_m L}\right] \quad \ldots (35)$$

$$b = \frac{\mu\rho}{B_0}\left[b_0 + e^{mt}\,.\,K_{\alpha_m}K_{\beta_m}\cdot\frac{\sinh\alpha_m Z\sinh\beta_m L - \sinh\alpha_m L\sinh\beta_m Z}{K_{\alpha_m}\sinh\alpha_m L\cosh\beta_m L - K_{\beta_m}\sinh\beta_m L\cosh\alpha_m L}\right] \quad (36)$$

where α_m and β_m are the values of α and β at $s = m$ when m is negative but $m > \dfrac{-\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}+1)^2}$.

The solution is the same as above.

(iii) $m < 0 \quad m < -\dfrac{\sigma B_0^2}{\rho(\sqrt{\mu\sigma\gamma}-1)^2}$.

Both the roots of (21) are real and negative
$\alpha^2 = -\alpha_1^2$, $\beta^2 = -\beta_1^2$ α_{1m} = value of α_1 at $s = m$ and β_{1m} = value of β_1 at $s = m$.

The solution will be

$$v = v_0 - f'(0)\,t + e^{mt}\left[1 - \frac{K_{\alpha_m} \sin \alpha_{1m}L \cos \beta_{1m}Z - K_{\beta_m} \sin \beta_{1m}L \cos \alpha_{1m}Z}{K_{\alpha_m} \sin \alpha_{1m}L \cos \beta_{1m}L - K_{\beta_m} \sin \beta_{1m}L \cos \alpha_{1m}L}\right] \quad (37)$$

$$b = \frac{\mu\rho}{B_0}\left[b_0 + e^{mt}\,.\,K_{\alpha_m}K_{\beta_m}\frac{\sin \alpha_{1m}Z \sin \beta_{1m}L - \sin \alpha_{1m}L \sin \beta_{1m}Z}{K_{\alpha_m} \sin \alpha_{1m}L \cos \beta_{1m}L - K_{\beta_m} \sin \beta_{1m}L \cos \alpha_{1m}L}\right]. \quad (38)$$

6. Conclusions

It has also been proved that when $\rho f'(t)$ = constant, the motion cannot become unsteady. This is the case studied by Ferraro, Cowling and Plumpton (1966) and Cowling (1957) in discussing the steady state. The time variation in pressure gradient together with external force creates unsteadiness in the flow.

If we analyse all the results of special case (ii), each solution for V consists of two terms, $V_0 - f'(0)\,t$ and $e^{mt}Q(z, m)$, while for b consists of

$$\frac{\mu\rho}{B_0}\,.\,b_0 \text{ and } \frac{\mu\rho}{B_0}\,e^{mt}Q'(m_1Z).$$

The terms $V_0 - f'(0)\,t$ and $\frac{\mu\rho}{B_0}\,b_0$ are the natural responses and the other terms are forced responses.

Forced responses depend on the driving force e^{mt} when m is positive, i.e. the driving force is exponentially increasing with time, the velocity and the magnetic component in its direction increases with time, $Q(m_1z)$ and $Q'(m_1z)$ are hyperbolic in Z.

When m is negative, i.e. the driving force is exponentially decreasing with time, the velocity and the magnetic component in its direction decreases with time. $Q(m_1z)$ is hyperbolic if $m > \frac{-\sigma\beta_0^2}{\rho(\sqrt{\mu\sigma\gamma}+1)^2}$ and is periodic if $m < \frac{-\sigma\beta_0^2}{\rho(\sqrt{\mu\sigma\gamma}-1)^2}$

Acknowledgement

The author's thanks are due to Dr. P. C. Vaidya, Professor of Mathematics, Gujarat University, for his help in the preparation of this paper.

References

Cowling, T. G. (1957). Magnetohydrodynamics. Interscience Publishers, Inc., New York, pp. 13, 14.

Ferraro, V. C. A., Cowling, T. G., and Plumpton, C. (1966). An Introduction to Magneto-fluid Mechanics (second edition). Oxford University Press, pp. 55–60.

Hartmann, J., and Lazarus, F. (1938). Hg. dynamics—II. *Det. Kgl. Denska Videsek*, 15, 7.