THERMAL STRESS IN A TWO-DIMENSIONAL INFINITE ELASTIC MEDIUM WEAKENED BY TWO COPLANAR GRIFFITH CRACKS

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Linear thermoelastic problem is solved for the thermal stress and displacement fields in a two-dimensional elastic medium of infinite extent weakened by two coplanar Griffith cracks. The faces of the cracks are heated by maintaining them at certain constant temperature. The problem is reduced to that of solving two sets of triple integral equations involving trigonometric kernels. These equations are solved by the application of finite Hilbert transform technique. The solutions are used for deriving expressions for temperature, displacement and stresses in the neighbourhood of crack in the closed form.

1. Introduction

In engineering practice, an important class of problems concerns the evaluation of the thermal stress set up in a heated elastic solid containing cracks. Because of the various technical applications particularly the importance of these problems in the theory of fractures, in recent years, considerable interest has been devoted to the problems of calculating thermal stresses in infinite, semi-infinite solids, thick slabs, infinite cylinders and elastic spheres containing penny-shaped cracks. Adequate references to these works are given in a paper by Srivastava and Palaiya (1969). Here we shall use the method of integral transform for studying the problem of finding the distribution of thermal stresses in an infinite two-dimensional elastic medium containing two Griffith cracks which are opened out by the application of prescribed temperature to their inner faces.

Formulation of Problem

We shall consider the temperature and displacement fields in perfectly elastic medium which is conducting heat. With regards to both its mechanical and thermal properties the medium is assumed to be isotropic and homogeneous. The stresses in the medium can be described by three components σ_{xx}, σ_{yy}, σ_{xy}. The displacement vector U may be taken to have the components $(u, v, 0)$. The two cracks in the medium occupy the region $-b < x < -a$, $a < x < b$, $y = 0$. We suppose that the cracks are opened out by application of prescribed temperature to their inner surfaces. Without loss of generality, the crack surfaces may be assumed to be free from mechanical loads, i.e. $\sigma_{yy}(x, 0) = 0$ for $a < |x| < b$. Since identical thermal and mechanical conditions are prescribed on the two surfaces of the cracks, the problem is equivalent to the problem of finding the distribution of thermal stress in a semi-infinite two-dimensional elastic medium $y \geqslant 0$ with the mixed boundary conditions on $y = 0$. In view of symmetry, the $y = 0$ axis must be free from shearing stress σ_{xy} and $v(x, 0)$ must vanish outside the regions occupied by the cracks. Thus the requisite thermal and elastic boundary conditions on $y = 0$ are taken to be

$$T(x) = -\sqrt{\frac{\pi}{2}} \cdot T_0 g(x); \quad a < |x| < b \quad \ldots \quad (1.1)$$

$$\frac{\partial T}{\partial y} = 0; \quad 0 < |x| < a, \; b < |x| < \infty \quad \ldots \quad (1.2)$$

$$\sigma_{xy}(x, 0) = 0; \quad 0 < |x| < \infty \quad \ldots \quad (1.3)$$

$$v(x, 0) = 0; \quad 0 < |x| < a, \; b < |x| < \infty \quad \ldots \quad (1.4)$$

$$\sigma_{yy}(x, 0) = 0; \quad a < |x| < b. \quad \ldots \quad (1.5)$$

In sections 2 and 3 the mixed boundary value problem posed by the equations (1.1–1.5) are first reduced to two sets of solvable triple integral equations with trigonometric kernels. The integral equations are solved by using the finite Hilbert transform technique given by Srivastava and Lowengrub (1970). The displacement and stress components for the physically important case, when the prescribed temperature is constant, are then expressed in the closed form. Another objective of this work is to calculate the stress intensity factor, the critical value of which controls the onset of crack propagation in brittle materials. In the Appendix we have quoted some results of Srivastava and Lowengrub (1970) which have been frequently used in the following sections.

2. Solution of the Two-dimensional Equations of Thermoelasticity

The equations of equilibrium of two-dimensional elastic medium conducting heat may be written in the forms (Nowacki 1962)

$$2(1-\eta)\frac{\partial^2 u}{\partial x^2} + (1-2\eta)\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} = 2\alpha_t(1+\eta)\frac{\partial T}{\partial x} \quad \ldots \quad (2.1)$$

$$(1-2\eta)\frac{\partial^2 v}{\partial x^2} + 2(1-\eta)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 2\alpha_t(1+\eta)\frac{\partial T}{\partial y} \quad \ldots \quad (2.2)$$

where T is the deviation of the absolute temperature of the elastic medium in a state of zero stress and strain, α_t the coefficient of linear expansion, η

Poisson's ratio and Y Young's modulus of elasticity. The temperature field is determined by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \qquad \text{(2.3)}$$

in the steady state and in the absence of thermal sources.

The partial differential equations (2.1) and (2.2) can be reduced to ordinary simultaneous linear differential equations by the introduction of Fourier sine and cosine transform. We define

$$\bar{u}(\xi, y) = F_s[u(x, y), x \to \xi] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty u(x, y) \sin \xi x \, dx \qquad \text{(2.4)}$$

$$\bar{v}(\xi, y) = F_c[v(x, y), x \to \xi] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty v(x, y) \cos \xi x \, dx \qquad \text{(2.5)}$$

$$\bar{T}(\xi, y) = F_c[T(x, y), x \to \xi]. \qquad \text{(2.6)}$$

If we multiply both the sides of (2.1) by $\sin \xi x$, integrate with respect to x from 0 to ∞ and make use of well-known properties of Fourier sine transform (Sneddon 1951), we get

$$[(1-2\eta)D^2-(2-2\eta)\xi^2]\bar{u}-\xi D\bar{v} = -2\alpha_t(1+\eta)\,\xi\bar{T} \qquad \text{(2.7)}$$

where $D \equiv \dfrac{d}{dy}$. Similarly if we multiply (2.2) and (2.3) by $\cos \xi x$ and integrate for all positive values of x we obtain

$$[(2-2\eta)D^2-(1-2\eta)\xi^2]\bar{v}+\xi D\bar{u} = 2\alpha_t(1+\eta)\,D\bar{T} \qquad \text{(2.8)}$$

$$[D^2-\xi^2]\bar{T} = 0. \qquad \text{(2.9)}$$

If we eliminate $\bar{u}$ and $\bar{v}$ from (2.7) and (2.8) in turn and make use of (2.9) we find that

$$(D^2-\xi^2)^2\bar{u} = 0, \quad (D^2-\xi^2)^2\bar{v} = 0. \qquad \text{(2.10)}$$

The boundary values appropriate to any given problem can be transformed to boundary values on the Fourier sine and cosine transforms $\bar{u}$, $\bar{v}$, $\bar{T}$ and eqns. (2.7) to (2.10) can be solved to give the values of these transforms. From these expressions we obtain the required solution by using the Fourier sine and cosine inversion theorems. Thus we get

$$u(x, y) = F_s[\bar{u}(\xi, y), \xi \to x] \qquad \text{(2.11)}$$

$$v(x, y) = F_c[\bar{v}(\xi, y), \xi \to x] \qquad \text{(2.12)}$$

$$T(x, y) = F_c[\bar{T}(\xi, y), \xi \to x]. \qquad \text{(2.13)}$$

3. Solutions for the Semi-infinite Two-dimensional Elastic Medium

In the case of semi-infinite elastic medium $y \geqslant 0$, assumed free from disturbances at infinity, we are interested in the solutions of (2.1) to (2.3) which tend to zero as $y \to \infty$. So that the appropriate solution of (2.3) is

$$\overline{T}(\xi, y) = A(\xi)\, e^{-\xi y} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where $A(\xi)$, which is a function of the parameter ξ, is determined from the boundary conditions on the temperature field. If we substitute this value of $\overline{T}(\xi, y)$ in eqns. (2.1) and (2.2), we get

$$[(1-2\eta)D^2-(2-2\eta)\xi^2]\bar{u}-\xi\, D\bar{v} = -2(1+\eta)\alpha_t\, \xi A(\xi)\, e^{-\xi y}$$

$$[(2-2\eta)D^2-(1-2\eta)\xi^2]\bar{v}+\xi\, D\bar{u} = -2(1+\eta)\alpha_t\, \xi A(\xi)\, e^{-\xi y}$$

which are easily seen to have the solutions

$$\bar{u} = (A_1+\xi B_1 y)\, e^{-\xi y}, \quad \bar{v} = (A_2+B_2\xi y)^{-\xi y} \qquad \ldots \quad \ldots \quad (3.2)$$

where

$$B_1 = B_2;\ A_1-A_2+(3-4\eta)B_2 = 2(1+\eta)\alpha_t\, \xi^{-1}A(\xi). \qquad \ldots \quad (3.3)$$

To find the further necessary relations among A_1, B_1, A_2 and B_2 we shall have to make use of the other boundary conditions. The shearing stress σ_{xy} is determined in terms of displacement vector by the equation

$$\sigma_{xy} = \mu\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]$$

where $\mu = Y/2(1+\eta)$ is the modulus of rigidity. If we multiply both the sides of this equation by $\sin \xi x$ and integrate with respect to x between 0 and ∞, we find that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sigma_{xy}(x, y)\sin \xi x\, dx = \mu(D\bar{u}-\xi\bar{v}) = -\mu\xi[A_1-B_1+A_2+(B_1+B_2)\xi y]\, e^{-\xi y}.$$

The boundary condition $\sigma_{xy} = 0$ on $y = 0$, therefore, implies that

$$A_1-B_1+A_2 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

From these equations we deduce

$$B = B_1 = B_2 = \frac{1}{2(1-\eta)}A_2+\frac{1+\eta}{2(1-\eta)}\alpha_t\, \xi^{-1}A(\xi) \qquad \ldots \quad \ldots \quad (3.5)$$

$$A_1 = -\frac{1-2\eta}{2(1-\eta)}A_2+\frac{1+\eta}{2(1-\eta)}\alpha_t\, \xi^{-1}A(\xi). \qquad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

Now from the stress-strain relations we know that

$$\sigma_{yy} = \frac{2\mu}{1-2\eta}\left[(1-\eta)\frac{\partial v}{\partial y}+\eta\frac{\partial u}{\partial x}-\alpha_t(1+\eta)T\right]$$

from which it follows that the cosine transform

$$\bar{\sigma}_{yy} = F_c[\sigma_{yy}(x, y),\ x \to \xi]$$

of the normal stress σ_{yy} is given in terms of $\bar{u}$, $\bar{v}$ and $\bar{T}$ by the equation

$$\bar{\sigma}_{yy} = \frac{2\mu}{1-2\eta}\left[(1-\eta)D\bar{v}+\eta\xi\bar{u}-\alpha_t(1+\eta)\bar{T}\right].$$

Substituting the values of $\bar{v}$, $\bar{u}$ and $\overline{T}$ we get

$$\bar{\sigma}_{yy} = -\frac{2\mu}{1-2\eta}\xi e^{-\xi y}[(1-\eta)(A_2-B)-\eta A_1+\alpha_t(1+\eta)\xi^{-1}A+(1-2\eta)B\,\xi y]$$

and making use of (3.5) and (3.6) we see that this is equivalent to the relation

$$\bar{\sigma}_{yy} = -\frac{\mu}{1-\eta}[A_2+\alpha_t(1+\eta)\xi^{-1}A](1+\xi y)\xi\, e^{-\xi y}. \quad \ldots \quad \ldots \quad (3.7)$$

Inverting the expressions (3.1) and (3.2) by Fourier inversion theorem we obtain the expressions

$$u(x,y) = \frac{1}{(2\pi)^{\frac{1}{2}}(1-\eta)}\int_0^\infty \left[(\xi y-1+2\eta)\,A_2(\xi)+\alpha_t(1+\eta)(1+\xi y)\frac{A(\xi)}{\xi}\right] \times e^{-\xi y}\sin\xi x\,d\xi \quad \ldots \quad (3.8)$$

$$v(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \left[A_2(\xi)+\frac{\xi y}{2(1-\eta)}(A_2(\xi)+\alpha_t(1+\eta)\xi^{-1}A(\xi))\right]e^{-\xi y}\cos\xi x\,d\xi \quad (3.9)$$

$$T(x,y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty A(\xi)\,e^{-\xi y}\cos\xi x\,d\xi; \quad \ldots \quad (3.10)$$

for the components of the displacement vector and the temperature. We note that for the solution for $y = 0$

$$\sigma_{xy} = 0$$

and

$$\sigma_{yy}(x,0) = -\frac{Y}{(2\pi)^{\frac{1}{2}}(1-\eta^2)}\int_0^\infty \xi\left[A_2(\xi)+\alpha_t(1+\eta)\frac{A(\xi)}{\xi}\right]\cos\xi x\,d\xi. \quad (3.11)$$

It is more convenient to write, $\alpha_t(1+\eta)\,A(\xi) = \dfrac{\phi(\xi)}{\xi}$ with this notation we find that the solution obtained above has the property that when $y = 0$

$$u(x,0) = \frac{1}{(2\pi)^{\frac{1}{2}}(1-\eta)}\int_0^\infty \left[\frac{\phi(\xi)}{\xi}-(1-2\eta)\,A_2(\xi)\right]\sin\xi x\,d\xi \quad \ldots \quad (3.12)$$

$$v(x,0) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty A_2(\xi)\cos\xi x\,d\xi \quad \ldots \quad (3.13)$$

$$T(x,0) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\alpha_t(1+\eta)}\int_0^\infty \frac{\phi(\xi)}{\xi}\cos\xi x\,d\xi \quad \ldots \quad (3.14)$$

$$\frac{\partial T}{\partial y} = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\alpha_t(1+\eta)}\int_0^\infty \frac{\phi(\xi)}{\xi}\cos\xi x\,d\xi \quad \ldots \quad (3.15)$$

$$\sigma_y(x,0) = -\frac{Y}{(2\pi)^{\frac{1}{2}}(1-\eta^2)}\int_0^\infty \xi\left(A_2(\xi)+\frac{\phi(\xi)}{\xi^2}\right)\cos\xi x\,d\xi \quad \ldots \quad (3.16)$$

$$= -\frac{Y}{(2\pi)^{\frac{1}{2}}(1-\eta^2)}\left[\frac{d}{dx}\int_0^\infty A_2(\xi)\sin\xi x\,d\xi+\int_0^\infty \frac{\phi(\xi)}{\xi}\cos\xi x\,d\xi\right]. \quad \ldots \quad (3.17)$$

The functions $A_2(\xi)$ and $\phi(\xi)$ can be determined if and only if we further specify the mechanical and thermal conditions on the boundary.

4. Reduction of the Problem to Two Sets of Triple Integral Equations with Trigonometric Kernel

We still have to satisfy the thermal conditions (1.1) and (1.2) and the mechanical conditions (1.4) and (1.5). The conditions (1.1) and (1.2) lead to the triple integral equations

$$\left.\begin{aligned} &\int_0^\infty \phi(\xi)\cos(\xi x)\,d\xi = 0, \quad 0<|x|<a \\ &\frac{2}{\pi}\int_0^\infty \frac{\phi(\xi)}{\xi}\cos \xi x\,d\xi = -T_0\alpha_t(1+\eta)\,g(x) = p(x), \quad a<|x|<b \\ &\int_0^\infty \phi(\xi)\cos(\xi x)\,d\xi = 0, \quad |x|>b \end{aligned}\right] \tag{A}$$

for the determination of the unknown function $\phi(\xi)$. Similarly the conditions (1.4) and (1.5) lead to the triple integral equations

$$\left.\begin{aligned} &\int_0^\infty A_2(\xi)\cos \xi x\,d\xi = 0, \quad 0<|x|<a \\ &\frac{2}{\pi}\int_0^\infty \xi A_2(\xi)\cos \xi x\,d\xi = -\frac{2}{\pi}\int_0^\infty \frac{\phi(\xi)}{\xi}\cos \xi x\,d\xi \\ &\qquad = T_0\alpha_t(1+\eta)\,g(x) = q(x), \quad a<|x|<b \\ &\int_0^\infty A_2(\xi)\cos(\xi x)\,d\xi = 0, \quad |x|>b. \end{aligned}\right] \tag{B}$$

Srivastava and Lowengrub (1970) had given a novel method, based on finite Hilbert transform, for solving this type of triple integral equations. The solutions given by Srivastava and Lowengrub (1970) are:
For the set of equations (A)

$$\phi(\xi) = \int_a^b \frac{h(t^2)}{t}(1-\cos \xi t)\,dt$$

where $h(t^2)$ is determined from the equation

$$h(t^2) = \frac{2}{\pi[(t^2-a^2)(b^2-t^2)]^{\frac{1}{2}}}\left[t^2\int_a^b \frac{[(b^2-y^2)(y^2-t^2)]^{\frac{1}{2}}}{y^2-t^2}\frac{d}{dy}(p(y))\,dy + \frac{2ab}{\log\frac{b-a}{b+a}}\int_a^b \frac{y\,p(y)\,dy}{[(y^2-a^2)(b^2-y^2)]^{\frac{1}{2}}}\right] \quad \ldots \quad (4.2)$$

provided $b>a>0$.

The solution for the set of equations (B) is

$$A_2(\xi) = \frac{1}{\xi}\int_a^b g(t^2)\sin \xi t\,dt \quad \ldots \quad (4.3)$$

where

$$g(t^2) = -\frac{2}{\pi}\int_a^b \left[\frac{(t^2-a^2)(b^2-t^2)}{(b^2-y^2)(y^2-a^2)}\right]^{\frac{1}{2}} \frac{y\,q(y)}{y^2-t^2}\,dy$$

$$+\frac{2b}{\pi F[(b^2-t^2)(t^2-a^2)]^{\frac{1}{2}}}\int_a^b\int_a^b \left[\frac{(x^2-a^2)(b^2-y^2)}{(b^2-x^2)(y^2-a^2)}\right]^{\frac{1}{2}} \frac{y\,q(y)}{y^2-x^2}\,dy\,dx. \quad \ldots \quad (4.4)$$

Once the nature of the prescribed temperature is known the functions $h(t^2)$ and $g(t^2)$ can be calculated from (4.2) and (4.4). Then the temperature and the thermoelastic fields can be evaluated from the following expressions for the temperature, the components of displacement and stress in terms of the functions $h(t^2)$ and $g(t^2)$

$$T(x, 0) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\alpha_t(1+\eta)}\int_a^b \frac{h(t^2)}{t}\log\left|1-\frac{t^2}{x^2}\right| dt \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

$$u(x, 0) = \frac{1}{2(2\pi)^{\frac{1}{2}}(1-\eta)}\int_a^b \left[\frac{h(t^2)}{t}\left\{x\log\left|1-\frac{t^2}{x^2}\right| + t\log\left|\frac{t+x}{t-x}\right|\right\}\right.$$

$$\left. -(1-2\eta)\,g(t^2)\log\left|\frac{t+x}{t-x}\right|\right] dt \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

$$v(x, 0) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_x^b g(t^2)\,dt, \quad a < x < b \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

$$\sigma_{yy}(x, 0) = -\frac{Y}{(2\pi)^{\frac{1}{2}}(1-\eta^2)}\left[\int_a^b \frac{t\,g(t^2)}{t^2-x^2}\,dt + \tfrac{1}{2}\int_a^b \frac{h(t^2)}{t}\log\left|1-\frac{t^2}{x^2}\right| dt\right]. \quad (4.8)$$

In deriving the above expressions we have used the results given in the Appendix.

5. The Prescribed Temperature is Constant

We now consider the case when the prescribed temperature is constant, i.e. $g(x) = 1$. Then

$$T(x, 0) = -T_0\left(\frac{\pi}{2}\right)^{\frac{1}{2}} = \theta_0, \quad a < |x| < b.$$

In this case it is easily shown that

$$h(t^2) = -\frac{2T_0\alpha_t(1+\eta)\,ab}{\log\dfrac{b-a}{b+a}}[(t^2-a^2)(b^2-t^2)]^{-\frac{1}{2}} \quad \ldots \quad \ldots \quad (5.1)$$

provided $b > a > 0$.

$$g(t^2) = \frac{\alpha_t T_0(1+\eta)(t^2-b^2E/F)}{[(t^2-a^2)(b^2-t^2)]^{\frac{1}{2}}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

The expressions for temperature, components of displacement and stress are

$$T(x,0) = -\theta_0 \begin{cases} \dfrac{2}{\log \dfrac{b-a}{b+a}} \log \dfrac{a\sqrt{b^2-x^2}+b\sqrt{a^2-x^2}}{x(a+b)}, & 0 < |x| \leqslant a \\ 1, & a \leqslant |x| \leqslant a \\ \dfrac{2}{\log \dfrac{b-a}{b+a}} \log \dfrac{a\sqrt{x^2-b^2}+b\sqrt{x^2-a^2}}{x(a+b)}, & b \leqslant |x| < \infty \end{cases} \quad \text{.. (5.3)}$$

$$u(x,0) = -\theta_0\alpha_t(1+\eta)\left[x + \frac{aK(a/b)}{2(1-\eta)\log \dfrac{b-a}{b+a}} + \frac{1-2\eta}{2-2\eta}\frac{b}{2F}\right], \quad a \leqslant x \leqslant b \quad \text{.. (5.4)}$$

$$v(x,0) = \frac{2\theta_0}{\pi}\alpha_t(1+\eta)\, b\left[E(\phi,q) - \frac{E}{F}(\phi,q)\right], \quad a \leqslant x \leqslant b \quad \text{.. .. .. (5.5)}$$

$$\sigma_{yy}(x,0) = -\frac{Y\theta_0\alpha_t}{2-2\eta}\begin{cases} \left[1 + \dfrac{x^2-b^2E/F}{[(b^2-x^2)(x^2-a^2)]^{\frac12}} - \dfrac{2}{\log \dfrac{b-a}{b+a}} \log \dfrac{a\sqrt{b^2-x^2}+b\sqrt{a^2-x^2}}{x(a+b)}\right], & 0 < x < a \\ 0, & a < x < b \\ 1 - \dfrac{x^2-b^2E/F}{[(x^2-b^2)(x^2-a^2)]^{\frac12}} - \dfrac{2}{\log \dfrac{b-a}{b+a}} \log \dfrac{a\sqrt{x^2-b^2}+b\sqrt{x^2-a^2}}{x(a+b)}, & b < x < \infty \end{cases} \quad (5.6)$$

where $\cos\phi = \left(\dfrac{b^2-y^2}{b^2-a^2}\right)^{\frac12}$, $q^2 = 1-\dfrac{a^2}{b^2}$, $E(\phi,q)$ and $F(\phi,q)$ are incomplete elliptic integrals and $E = E\left(\dfrac{\pi}{2},q\right)$, $F = F\left(\dfrac{\pi}{2},q\right)$ are complete elliptic integrals of first and second kind and

$$K(a/b) = F(\pi/2, a/b).$$

These expressions are derived with the help of integrals given in the Appendix.

The stress concentration factor at the edges of the crack is defined by the equations

$$N_a = \mathrm{Lt}_{x\to a-}(a-x)^{\frac12}[\sigma_{yy}(x,0)], \quad 0 < x < a$$
$$N_b = \mathrm{Lt}_{x\to b+}(x-b)^{\frac12}[\sigma_{yy}(x,0)], \quad b < x < \infty.$$

Hence

$$N_a = \frac{YT_0\alpha_t(\pi/2)^{\frac12}}{2-2\eta}\cdot\frac{b^2E/F-a^2}{[2a(b^2-a^2)]^{\frac12}} \quad \text{.. .. .. (5.7)}$$

$$N_b = \frac{YT_0\alpha_t(\pi/2)^{\frac12}}{2-2\eta}\cdot\frac{b^2(1-E/F)}{[2b(b^2-a^2)]^{\frac12}}. \quad \text{.. .. .. (5.8)}$$

Another interesting fact to note is that there is a concentration of temperature and thermal stresses at origin. Damage in elastic medium due to this fact can be avoided by creating a small hole at the origin.

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Appendix

Here we quote results from Srivastava and Lowengrub (1970) that we have frequently used

$$\int_0^\infty \frac{\sin \xi x \sin \xi t}{\xi}\, d\xi = \tfrac{1}{2} \log \left| \frac{x+t}{x-t} \right|$$

$$\int_0^\infty \frac{\cos \xi x(1-\cos \xi t)}{\xi}\, d\xi = \tfrac{1}{2} \log \left| 1-\frac{t^2}{x^2} \right|$$

$$\int_0^\infty \frac{\sin \xi x(1-\cos \xi t)}{\xi^2}\, d\xi = \tfrac{1}{2}x \log \left| 1-\frac{t^2}{x^2} \right| + \frac{t}{2} \log \left| \frac{t+x}{t-x} \right|$$

$$\int_a^b \frac{t\, dt}{\sqrt{(t^2-a^2)(b^2-t^2)(t^2-x^2)}} = \begin{cases} \frac{\pi}{2} [(a^2-x^2)(b^2-x^2)]^{\frac{1}{2}} & ;\ 0 < y < a \\ 0 & ;\ a < y < b \\ -\frac{\pi}{2} [(x^2-a^2)(x^2-b^2)]^{\frac{1}{2}} & ;\ b < y < \infty \end{cases}$$

$$\int_a^b \frac{\log \left| 1-\frac{t^2}{x^2} \right|}{t\sqrt{(t^2-a^2)(b^2-t^2)}}\, dt = \begin{cases} \frac{\pi}{ab} \log \frac{a\sqrt{b^2-x^2}+b\sqrt{a^2-x^2}}{x(a+b)} & ;\ 0 < x \leqslant a \\ \frac{\pi}{2ab} \log \frac{b-a}{b+a} & ;\ a < x < b \\ \frac{\pi}{ab} \log \frac{a\sqrt{x^2-b^2}+b\sqrt{x^2-a^2}}{x(a+b)} & ;\ b \leqslant x < \infty \end{cases}$$

$$\int_a^b \frac{\log \left| \frac{t+x}{t-x} \right|}{[(t^2-a^2)(b^2-t^2)]^{\frac{1}{2}}}\, dt = \begin{cases} \frac{\pi}{b} F\left(\sin^{-1} \frac{y}{a}, \frac{a}{b}\right) & ;\ 0 < y \leqslant a \\ \frac{\pi}{b} K\left(\frac{a}{b}\right) & ;\ a < x < b \\ \frac{\pi}{b} F\left(\sin^{-1} \frac{b}{y}, \frac{a}{b}\right) & ;\ y \geqslant b \end{cases}$$

$$\int_a^b \frac{(t^2-b^2E/F) \log \left| \frac{t+x}{t-x} \right|}{[(t^2-a^2)(b^2-t^2)]^{\frac{1}{2}}}\, dt = \pi\left(x-\frac{\pi b}{2F}\right);\ a < x < b.$$