ON A GENERALIZATION OF HANKEL TRANSFORM AND SELF-RECIPROCAL FUNCTIONS

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In this paper the authors have given a generalization of the Hankel transform in which the Meijer G-function plays the role of being a symmetrical Fourier kernel. Several properties of functions self-reciprocal under this transform analogous to those given by Hardy and Titchmarsh have been derived therefrom. The results arrived at have been supported by means of examples.

§ 1. A function $K(x)$ by means of which an arbitrary function $f(x)$ subjected to appropriate conditions is capable of being represented as a repeated integral

$$f(x) = \int_0^\infty \int_0^\infty K(xu)\, K(uy) f(y)\, dy\, du \quad \ldots \quad (1.1)$$

has been called a symmetrical Fourier kernel. Usually (1.1) is written as a pair of reciprocal integral equations

$$g(x) = \int_0^\infty K(xy)\, f(y)\, dy \quad \ldots \quad (1.2)$$

and

$$f(x) = \int_0^\infty K(xy)\, g(y)\, dy. \quad \ldots \quad (1.3)$$

In an earlier paper the authors (Kapoor and Masood 1968) have proved the theorem:

If

$$\phi(s) = \int_0^\infty G^{m,n}_{p,q}\left(\alpha s^\lambda t^\lambda \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) f(t)\, dt, \quad \ldots \quad (1.4)$$

then

$$f(t) = \frac{\lambda}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=m+1}^{q} \Gamma\left(1-b_j+\frac{k-1}{\lambda}\right) \prod_{j=n+1}^{p} \Gamma\left(a_j+\frac{1-k}{\lambda}\right)}{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{1-k}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)} \alpha^{\frac{1-k}{\lambda}} t^{-k} \int_0^\infty s^{-k} \phi(s)\, ds\, dk \quad \ldots \quad (1.5)$$

under suitable conditions stated therein.

From (1.5), on inverting the order of integration, a process which is easily justifiable, we have

$$f(t) = \frac{\lambda}{2\pi i}\int_0^\infty \phi(s)\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\frac{k-1}{\lambda}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{1-k}{\lambda}\right)}{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{1-k}{\lambda}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\frac{k-1}{\lambda}\right)}$$
$$\times \alpha^{\frac{1-k}{\lambda}}(st)^{-k}dk\, ds. \quad \ldots \quad (1.6)$$

Under certain restrictions the inner integral in (1.6) can be expressed as a Meijer's G-function, so that (1.4) and (1.6) together form a pair of reciprocal integral equations. The object, therefore, of this paper is to reduce the two G-functions involved as kernels of transformation in (1.4) and (1.6) to a single-G-function, which will be capable of serving as a symmetrical Fourier kernel in accordance with the integral equations (1.2) and (1.3). Further, a brief study of the self-reciprocal functions associated with this G-function as a symmetrical Fourier kernel has also been made.

§ 2. On replacing p and q by $2p$ and $2q$ respectively and substituting q for m and p for n, making a little simplification and using the result of Erdélyi (1954*b*, p. 207), (1.4) and (1.6) reduce to

$$\phi(s) = \int_0^\infty G^{q,\,p}_{2p,\,2q}\left(\alpha s^\lambda t^\lambda \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right) f(t)\, dt \quad \ldots \quad (2.1)$$

and

$$f(t) = \alpha^{\frac{1}{\lambda}}\lambda^2 \int_0^\infty G^{q,\,p}_{2p,\,2q}\left(\alpha s^\lambda t^\lambda \,\middle|\, \begin{matrix} 1-\frac{1}{\lambda}-(a_{p+1,\,2p}),\ 1-\frac{1}{\lambda}-(a_p) \\ 1-\frac{1}{\lambda}-(b_{q+1,\,2q}),\ 1-\frac{1}{\lambda}-(b_q) \end{matrix}\right)\phi(s)\, ds \quad (2.2)$$

respectively, where $(a_r) = a_1, a_2, \ldots, a_r$ and $(a_{r+1,\,t}) = a_{r+1}, a_{r+2}, \ldots, a_t$.

Consequently, on replacing $f(t)$ by $\lambda\,\alpha^{\frac{1}{2\lambda}} f(t)$ in (2.1) and (2.2) and assuming

$$a_i + a_{p+i} = 1-\frac{1}{\lambda} \quad \text{and} \quad b_j + b_{q+j} = 1-\frac{1}{\lambda}$$

for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, we finally arrive at the following:

If

$$\phi(s) = \int_0^\infty K(st)f(t)\, dt, \quad \ldots \quad (2.3)$$

then

$$f(t) = \int_0^\infty K(st)\,\phi(s)\, ds, \quad \ldots \quad (2.4)$$

where

$$K(x) = \lambda\alpha^{\frac{1}{2\lambda}} G^{q,\,p}_{2p,\,2q}\left(\alpha x^{\lambda}\,\middle|\,\begin{matrix}(a_p),\ 1-\frac{1}{\lambda}-(a_p)\\[2mm] (b_q),\ 1-\frac{1}{\lambda}-(b_q)\end{matrix}\right). \qquad \ldots \quad \ldots \quad (2.5)$$

Thus it follows that the function $K(x)$ as given by (2.5) is a symmetrical Fourier kernel. This result takes the form, which has otherwise been obtained by Fox (1961) by an entirely different process on taking $\lambda = \frac{1}{c}$ and $\alpha^{\frac{1}{\lambda}} = k$.

It is easy to deduce (Titchmarsh 1948, p. 220) that

$$\sqrt{\beta}\,K(\beta x) \text{ and } \gamma x^{\frac{1}{2}(\ \ -1)}\,K(x^{\gamma})$$

are also symmetrical Fourier kernels where $K(x)$ is given by (2.5). The reciprocity between (2.3) and (2.4) has rigorously been established by Fox. We therefore, proceed with the further study of the said transform.

Further, on specializing the parameters in the symmetrical Fourier kernel (2.5) it is easy to arrive at the Fourier kernels given earlier by Roop Narain (1956–57, 1959) and Mehra (1958).

A Transform Pair

From (2.3), (2.4) and (2.5) we obtain on making use of a known result (Erdélyi 1954*a*, p. 337) the following pair of the G-transform:

$$s^{\mu-\frac{1}{2}} \text{ and } \lambda^{\mu} s^{-\mu-\frac{1}{2}} \frac{\prod_{j=1}^{q}\Gamma\left(b_j+\frac{\mu}{\lambda}+\frac{1}{2\lambda}\right)\prod_{j=1}^{p}\Gamma\left(1-a_j-\frac{\mu}{\lambda}-\frac{1}{2\lambda}\right)}{\prod_{j=1}^{q}\Gamma\left(b_j-\frac{\mu}{\lambda}+\frac{1}{2\lambda}\right)\prod_{j=1}^{p}\Gamma\left(1-a_j+\frac{\mu}{\lambda}-\frac{1}{2\lambda}\right)},$$

where

$$-\min_{1\leqslant j\leqslant q}\operatorname{Re}(b_j) < \operatorname{Re}\left(\frac{\mu}{\lambda}+\frac{1}{2\lambda}\right) < 1-\max_{1\leqslant j\leqslant p}\operatorname{Re}(a_j).$$

§ 3. In this section self-reciprocal functions associated with $K(x)$ as given by (2.5) are deduced. If

$$f(x) = \int_0^{\infty} K(xy)\,f(y)\,dy, \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

then $f(x)$ is said to be the self-reciprocal function for the kernel $K(x)$. All symmetrical Fourier kernels can be associated with self-reciprocal functions and conversely.

We shall now determine conditions under which a function can be a solution of the integral equation (3.1), where $K(x)$ is the kernel (2.5), so that it may be self-reciprocal for the kernel $K(x)$.

Let $F(s)$ be the Mellin transform of $f(x)$. Then

$$F(s) = \int_0^\infty x^{s-1} f(x)\,dx \qquad (s = \sigma + it)$$

$$= \lambda\alpha^{\frac{1}{2\lambda}} \int_0^\infty x^{s-1} \int_0^\infty G^{q,p}_{2p,2q}\left(\alpha x^\lambda y^\lambda \middle| \begin{matrix} (a_p),\, 1-\frac{1}{\lambda}-(a_p) \\ (b_q),\, 1-\frac{1}{\lambda}-(b_q) \end{matrix}\right) f(y)\,dy\,dx$$

$$= \lambda\alpha^{\frac{1}{2\lambda}} \int_0^\infty f(y)\,dy \int_0^\infty x^{s-1} G^{q,p}_{2p,2q}\left(\alpha x^\lambda y^\lambda \middle| \begin{matrix} (a_p),\, 1-\frac{1}{\lambda}-(a_p) \\ (b_q),\, 1-\frac{1}{\lambda}-(b_q) \end{matrix}\right) dx$$

$$= \alpha^{\frac{1-2s}{2\lambda}} \int_0^\infty y^{-s} f(y)\,dy \int_0^\infty u^{\frac{s}{\lambda}-1} G^{q,p}_{2p,2q}\left(u \middle| \begin{matrix} (a_p),\, 1-\frac{1}{\lambda}-(a_p) \\ (b_q),\, 1-\frac{1}{\lambda}-(b_q) \end{matrix}\right) du$$

$$= \alpha^{\frac{1-2s}{2\lambda}} \frac{\prod_{j=1}^{q} \Gamma\left(b_j+\frac{s}{\lambda}\right) \prod_{j=1}^{p} \Gamma\left(1-a_j-\frac{s}{\lambda}\right)}{\prod_{j=1}^{q} \Gamma\left(\frac{1}{\lambda}+b_j-\frac{s}{\lambda}\right) \prod_{j=1}^{p} \Gamma\left(1-\frac{1}{\lambda}-a_j+\frac{s}{\lambda}\right)} F(1-s)$$

on using a known integral relation (1), provided that

$$-\min_{1\leqslant j\leqslant q} \operatorname{Re}(b_j) < \operatorname{Re}\left(\frac{s}{\lambda}\right) < 1 - \max_{1\leqslant j\leqslant p} \operatorname{Re}(a_j).$$

The inversion of the order of integration can easily be justified by De la Vallee Poussin's theorem, provided that (3.1) is absolutely convergent and the Mellin transform of $|f(x)|$ exists. Now, on putting

$$F(s) = \alpha^{-\frac{s}{2\lambda}} \frac{\prod_{j=1}^{q} \Gamma\left(b_j+\frac{s}{\lambda}\right)}{\prod_{j=1}^{p} \Gamma\left(1-\frac{1}{\lambda}-a_j+\frac{s}{\lambda}\right)} \psi(s),$$

we get a functional relation

$$\psi(s) = \psi(1-s) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

and, then, by Mellin's inversion formula (Titchmarsh 1948)

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \alpha^{-\frac{s}{2\lambda}} \frac{\prod_{j=1}^{q} \Gamma\left(b_j+\frac{s}{\lambda}\right)}{\prod_{j=1}^{p} \Gamma\left(1-\frac{1}{\lambda}-a_j+\frac{s}{\lambda}\right)} x^{-s}\psi(s)\,ds,$$

where $\psi(s)$ satisfies the functional relation (3.2).

By proceeding on similar lines as given by Titchmarsh (1948, p. 252) for the Hankel transform, it is easy to arrive at the following:

A necessary and sufficient condition that a function $f(x)$ of $A(\beta, b)$ (in the sense of Titchmarsh (1948, p. 252)) be self-reciprocal for the kernel given in (2.5) is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \alpha^{-\frac{s}{2\lambda}} \frac{\prod_{j=1}^{q} \Gamma\left(b_j+\frac{s}{\lambda}\right)}{\prod_{j=1}^{p} \Gamma\left(1-\frac{1}{\lambda}-a_j+\frac{s}{\lambda}\right)} x^{-s}\psi(s)\, ds \qquad .. \quad (3.3)$$

where $\psi(s)$ is regular, satisfies the equation (3.2) in the strip

$$b < \sigma < 1-b\,; \qquad .. \quad .. \quad .. \quad .. \quad (3.4)$$

$$\psi(s) = 0\left\{e^{(q-p)\left(\frac{\pi}{2\lambda}-\beta+\eta\right)|t|}\right\}$$

for every positive η and uniformly in any strip interior to (3.4) and c is any value of σ in (3.4).

Corollary—On taking $\alpha = \frac{1}{4}$ and $\lambda = 2$, $p = 1$, $q = 2$, $a_1 = k-m-\frac{1}{2}\nu-\frac{1}{4}$, $b_1 = \frac{1}{2}\nu+\frac{1}{4}$ and $b_2 = \frac{1}{2}\nu+2m+\frac{1}{4}$, we arrive at a result given by Roop Narain (6, p. 284) for the $\chi_{\nu, k, m}$—transform.

Example

The Mellin transform (Erdélyi 1954*a*, p. 337) of $G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right)$ is

$$\frac{\prod_{j=1}^{m} \Gamma(b_j+s) \prod_{j=1}^{n} \Gamma(1-a_j-s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j-s) \prod_{j=n+1}^{p} \Gamma(a_j+s)}, \qquad .. \quad .. \quad .. \quad (3.5)$$

where $p+q < 2\ (m+n)$, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$ and $-\min_{1\leqslant j\leqslant m} \mathrm{Re}\ (b_j) < \mathrm{Re}\ (s) < 1-\max_{1\leqslant j\leqslant n} \mathrm{Re}\ (a_j)$.

Therefore, on applying Mellin's inversion formula and thereafter replacing x by $\alpha^{\frac{1}{2}}x^{\lambda}$ and s by $\frac{s}{\lambda}$, we get

$$G_{r+p,\,h+q}^{m+q,\,n}\left(\alpha^{\frac{1}{2}}x^{\lambda}\,\middle|\,\begin{matrix}(a_r),\ 1-\frac{1}{\lambda}-(a_p)\\(b_q),\ (b_h)\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \alpha^{-\frac{s}{2\lambda}} \frac{\prod_{i=1}^{q} \Gamma\left(b_i+\frac{s}{\lambda}\right) \prod_{j=1}^{m} \Gamma\left(b_j+\frac{s}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j-\frac{s}{\lambda}\right)}{\prod_{i=1}^{p} \Gamma\left(1-\frac{1}{\lambda}-a_i+\frac{s}{\lambda}\right) \prod_{j=m+1}^{h} \Gamma\left(1-b_j-\frac{s}{\lambda}\right) \prod_{j=n+1}^{r} \Gamma\left(a_j+\frac{s}{\lambda}\right)} x^{-s}\frac{ds}{\lambda}.$$

This function can now be self-reciprocal if

$$\psi(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{s}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j-\frac{s}{\lambda}\right)}{\lambda \prod_{j=m+1}^{h} \Gamma\left(1-b_j-\frac{s}{\lambda}\right) \prod_{j=n+1}^{r} \Gamma\left(a_j+\frac{s}{\lambda}\right)}$$

satisfies the functional relation $\psi(s) = \psi(1-s)$, i.e. if

$$\frac{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{s}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j-\frac{s}{\lambda}\right)}{\prod_{j=m+1}^{h} \Gamma\left(1-b_j-\frac{s}{\lambda}\right) \prod_{j=n+1}^{r} \Gamma\left(a_j+\frac{s}{\lambda}\right)} = \frac{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{1}{\lambda}-\frac{s}{\lambda}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j-\frac{1}{\lambda}+\frac{s}{\lambda}\right)}{\prod_{j=m+1}^{h} \Gamma\left(1-b_j-\frac{1}{\lambda}+\frac{s}{\lambda}\right) \prod_{j=n+1}^{r} \Gamma\left(a_j+\frac{1}{\lambda}-\frac{s}{\lambda}\right)}.$$

This relation admits a solution if $m = n$, $r = h$ and $a_j+b_j = 1-\frac{1}{\lambda}$, for $j = 1, 2, \ldots, h$. Hence, we have

$$G^{n+q,\, n}_{h+p,\, h+q}\left(\alpha^{\frac{1}{2}} x^{\lambda} \,\middle|\, \begin{matrix} (a_h),\ 1-\frac{1}{\lambda}-(a_p) \\ (b_q),\ 1-\frac{1}{\lambda}-(b_h) \end{matrix}\right) \quad \ldots \quad \ldots \quad (3.6)$$

as a self-reciprocal function for the kernel $K(x)$ given by (2.5).

Particular Case

When $\alpha = \frac{1}{4}$, $\lambda = 2 = q$, $p = 1$, $a_1 = k-m-\frac{1}{2}\nu-\frac{1}{4}$, $b_1 = \frac{1}{2}\nu+\frac{1}{4}$ and $b_2 = 2m+\frac{1}{2}\nu+\frac{1}{4}$, we find that the function

$$G^{n+2,\, n}_{h+1,\, h+2}\left(\tfrac{1}{2}x^2 \,\middle|\, \begin{matrix} (a_h),\ \frac{1}{2}\nu+m-k+\frac{3}{4} \\ \frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}\nu+2m+\frac{1}{4},\ \frac{1}{2}-(a_h) \end{matrix}\right) \quad \ldots \quad \ldots \quad (3.7)$$

is self-reciprocal for $\chi_{\nu,k,m}$—transform, a result given by Roop Narain (1956–57, p. 286).

Several other functions self-reciprocal for other generalizations of Hankel transform may be deduced as particular cases of (3.6).

§ 4. The function $f(x)$ satisfying the integral equation (3.1), where $K(x)$ is the kernel given in (2.5), shall for brevity be denoted by $R(a_p, b_q)$, i.e. self-reciprocal in the generalized transform associated with the G-function (2.5) as the kernel.

We proceed to give below some rules connecting different classes of self-reciprocal functions. The proofs of these rules can be developed on the same lines as in the proofs of the corresponding rules of Hankel transform (Titchmarsh 1948, pp. 268–70). Further we assume that $f(x)$ is integrable in $(0, \infty)$ and the integrals involved exist.

Rule 1—If

$f(x)$ is $R(a_p, b_q)$ and

$$P(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \alpha^{-\frac{s}{\lambda}} \frac{\prod_{j=1}^{q}\Gamma\left(b_j+\frac{s}{\lambda}\right)\prod_{i=1}^{q}\Gamma\left(\delta_i+\frac{s}{\lambda}\right)}{\prod_{j=1}^{p}\Gamma\left(1-\frac{1}{\lambda}-a_j+\frac{s}{\lambda}\right)\prod_{i=1}^{p}\Gamma\left(1-\frac{1}{\lambda}-\rho_i+\frac{s}{\lambda}\right)} x^{-s}\omega(s)\, ds, \quad \ldots \quad (4.1)$$

where

$$\omega(s) = \omega(1-s), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

then

$$g(x) = \int_0^\infty P(xy)f(y)\,dy \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

is $R(\rho_p, \delta_q)$.

This rule corresponds to rule 2 of Titchmarsh (1948, p. 268). Also, since the rule is symmetrical in a, ρ and b, δ a kernel $P(x)$ which transforms $R(a_p, b_q)$ into $R(\rho_p, \delta_q)$ also effects the converse transformation.

Example

Taking $\omega(s) = \frac{1}{\lambda}$ and using the Mellin transform, we obtain

$$P(x) = G_{2p,\,2q}^{2q,\,0}\left(\alpha x^{\lambda}\,\middle|\,\begin{matrix}1-\frac{1}{\lambda}-(a_p),\ 1-\frac{1}{\lambda}-(\rho_p)\\ (b_q),\ (b_\delta)\end{matrix}\right) \quad \ldots \quad \ldots \quad (4.4)$$

as a kernel transforming $R(a_p, b_q)$ into $R(\rho_p, \delta_q)$ and vice versa.

The corresponding result due to Roop Narain (1956–57) can be obtained as a particular case of (4.4).

Rule 2—If

$f(x)$ is $R(a_p, b_q)$ and

$$P(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{q}\Gamma\left(b_j+\frac{s}{\lambda}\right)\prod_{i=1}^{q}\Gamma\left(\delta_i+\frac{1}{\lambda}-\frac{s}{\lambda}\right)}{\prod_{j=1}^{p}\Gamma\left(1-\frac{1}{\lambda}-a_j+\frac{s}{\lambda}\right)\prod_{j=1}^{p}\Gamma\left(1-\rho_i-\frac{s}{\lambda}\right)} x^{-s}w(s)\, ds, \quad (4.5)$$

where

$$\omega(s) = \omega(1-s),$$

then

$$g(x) = \frac{1}{x}\int_0^\infty P\left(\frac{y}{x}\right)f(y)\,dy \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

is $R(\rho_p, \delta_q)$.

This result corresponds to rule 3 of Titchmarsh (1948, p. 270).

Example

Taking $\omega(s) = \frac{1}{\lambda}$ and using the Mellin transform (3.5), we obtain

$$G^{q,\,q}_{p+q,\,p+q}\left(x^{\lambda}\,\middle|\,\begin{matrix}1-\frac{1}{\lambda}-(\delta_q),\ 1-\frac{1}{\lambda}-(a_p)\\ (b_q),\ (\rho_p)\end{matrix}\right)$$

as a kernel transforming $R(a_p,\ b_q)$ into $R(\rho_p,\ \delta_q)$.

Particular Case

On taking $\lambda = 2 = q,\ p = 1,\ \delta_1 = \frac{1}{2}\mu-\frac{1}{4},\ a_1 = k-m-\frac{1}{2}\nu-\frac{1}{4},\ b_1 = \frac{1}{2}\nu+\frac{1}{4},\ b_2 = \frac{1}{2}\nu+2m+\frac{1}{4},\ \delta_2 = \frac{1}{2}\mu+2n-\frac{1}{4}$ and $\rho_1 = l-n-\frac{1}{2}\mu-\frac{1}{4}$, we get the result due to Roop Narain (1956–57), namely

$$G^{22}_{33}\left(x^2\,\middle|\,\begin{matrix}\frac{1}{4}-\frac{1}{2}\mu,\ \frac{1}{4}-\frac{1}{2}\mu-2n,\ m-k+\frac{1}{2}\nu+\frac{3}{4}\\ \frac{1}{2}\nu+\frac{1}{4},\ \frac{1}{2}\nu+2m+\frac{1}{4},\ l-n-\frac{1}{2}\mu-\frac{1}{4}\end{matrix}\right) \qquad \cdots \quad (4.7)$$

as a kernel transforming $R_\nu(k,\ m)$ into $R_\mu(l,\ n)$.

Rule 3—If

$f(x)$ is $R(a_p,\ b_q)$ and $P(x)$ satisfies the relation

$$x\,P(x) = P\left(\frac{1}{x}\right), \qquad \cdots \quad (4.8)$$

then

$$g(x) = \frac{1}{x}\int_0^\infty P\left(\frac{y}{x}\right) f(y)\,dy \qquad \cdots \quad (4.9)$$

is also $R(a_p,\ b_q)$, where $P(x)$ is the same as in Rule 2 with $\rho_p \equiv a_p$ and $\delta_q \equiv b_q$. This result corresponds to Rule 4 of Titchmarsh (1948, p. 270).

Example

(i) Let $P(x) = x^{\frac{1}{2}(mn-1)}\,(x^m+1)^{-n}$.

Then, if $f(x)$ is $R(a_p,\ b_q)$, so also is

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}(mn-1)}\ (x^m+y^m)^{-n}\,f(y)\,dy.$$

(ii) Let $P(x) = x^{-\frac{1}{2}}\,F(x)$, where $F(x) = F\left(\frac{1}{x}\right)$.

Then, if $F(x)$ is $R(a_p, b_q)$, so also is

$$g(x) = \int_0^\infty (xy)^{-\frac{1}{2}} F\left(\frac{y}{x}\right) f(y)\, dy.$$

Taking $f(x) = G^{n+q,\, n}_{h+p,\, h+q}\left(\alpha^{\frac{1}{2}} x^{\lambda} \middle| \begin{matrix} (a_h),\ 1-\frac{1}{\lambda}-(a_p) \\ (b_q),\ 1-\frac{1}{\lambda}-(a_h) \end{matrix} \right)$

which is $R(a_p, b_q)$, we get

$$g(x) = \int_0^\infty \omega^{-\frac{1}{2}} G^{n+q,\, n}_{h+p,\, h+q}\left(\alpha^{\frac{1}{2}} \omega^{\lambda} x^{\lambda} \middle| \begin{matrix} (a_h),\ 1-\frac{1}{\lambda}-(a_p) \\ (b_q),\ 1-\frac{1}{\lambda}-(a_h) \end{matrix} \right) F(\omega)\, d\omega$$

to be a $R(a_p, b_q)$ function, if $F(\omega) = F\left(\frac{1}{\omega}\right)$.

In particular, when $p = 1$, $q = 2$, $\lambda = 2$, $\alpha = \frac{1}{4}$, $n = 0 = h$, $a_1 = k-m-\frac{\nu}{2} = -\frac{1}{4}$, $b_1 = \frac{1}{2}\nu+\frac{1}{4}$ and $b_2 = 2m+\frac{1}{2}\nu+\frac{1}{4}$, then on using the result of Erdélyi (1954*b*, p. 221 (68)), we get a result due to Roop Narain (6).

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