ON FINITE EXPANSIONS FOR THE H-FUNCTION

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In this paper we establish seven finite expansions for the H-function. On account of a most general character of the H-function, expansions for other special functions occurring in physics and applied mathematics can be obtained as special cases of our results. We record here finite expansions for Meijer's G-function and hypergeometric functions. The particular cases of these expansions yield various results recently obtained by Bose, Pathan, Kulshreshtha and others.

1. Introduction

The H-function of Fox (1961, p. 408) is defined and expressed as follows:

$$H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right]=H_{p,q}^{m,n}\left[x\,\middle|\,\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right]$$

$$=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s\,ds \qquad \ldots \quad (1.1)$$

where x is not equal to zero and an empty product is interpreted as unity; p, q, m, n are integers satisfying $1 < m < q$; $0 < n < p$; α_j $(j=1,\ldots,p)$; β_j $(j=1,\ldots,q)$ are positive numbers and a_j $(j=1,\ldots,p)$, b_j $(j=1,\ldots,q)$ are complex numbers such that no poles of $\Gamma(b_h-\beta_h s)$ $(h=1,\ldots,m)$ coincide with any pole of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,\ldots,n)$; i.e.

$$\alpha_i(b_h+\nu)\neq(a_i-\eta-1)\beta_h \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

$$(\nu,\eta=0,1,\ldots);\ (h=1,\ldots,m;\ i=1,\ldots,n).$$

Further the contour L runs from $\sigma-i\infty$ to $\sigma+i\infty$ such that the points

$$s=(b_h+\nu)/\beta_h\ \ (h=1,\ldots,m;\ \nu=0,1,\ldots) \qquad \ldots \quad \ldots \quad (1.3)$$

which are poles of $\Gamma(b_h-\beta_h s)$ lie to the right of L and the points

$$s=(a_i-\eta-1)/\alpha_i\ \ (i=1,\ldots,n;\ \eta=0,1,\ldots) \qquad \ldots \quad \ldots \quad (1.4)$$

which are poles of $\Gamma(i-a_i+\alpha_i s)$ lie to the left of L. Such a contour is possible on account of (1.2).

2. Finite Expansions for the H-function

The following expansions have been obtained in this section:

$$\sum_{r=0}^{k} k_{c_r}(1+a_1-b_q)_r H_{p,q}^{m,n}\left[z\left|\begin{matrix}\left(a_1-\frac{k}{2}+r,\alpha_1\right),\left(-a_1-\frac{k}{2}-1,\alpha_1\right),(a_3,\alpha_3),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_{q-1},\beta_{q-1}),\left(b_q-1-\frac{k}{2},\alpha_1\right)\end{matrix}\right.\right]$$

$$=H_{p,q}^{m,n}\left[z\left|\begin{matrix}\left(a_1+\frac{k}{2},\alpha_1\right),\left(-a_1-1-\frac{k}{2},\alpha_1\right),(a_3,\alpha_3),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_{q-1},\beta_{q-1}),\left(b_q+\frac{k}{2}-1,\alpha_1\right)\end{matrix}\right.\right] \quad .. \quad (2.1)$$

where

$$p \geqslant n \geqslant 2,\ q \geqslant 1,\ \mathrm{Re}\left(a_1-\frac{k}{2}-\sigma\alpha_1\right)\neq 0,-1,-2,\ldots,-(k-1).$$

$$\sum_{r=0}^{k} k_{c_r}(1-\alpha)_r H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+\beta-1,\alpha_1)\\(a_1+\alpha+\beta-r-2,\alpha_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+\beta-k-1,\alpha_1)\\(a_1+\alpha+\beta-k-2,\alpha_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)\end{matrix}\right.\right] \quad .. \quad (2.2)$$

where

$$p>n\geqslant 0,\ q\geqslant m\geqslant 1,\ \mathrm{Re}\,(3-a_1-\alpha-\beta-\alpha_1\sigma)\neq 0,-1,-2,\ldots,-(k-1).$$

$$\sum_{r=0}^{k} k_{c_r}\Gamma(c-\beta-r)H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+c-1,\alpha_1)\\(a_1+\alpha-1,\alpha_1),(a_1+\beta+r-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$=\Gamma(c-\beta-k)H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+c-k-1,\alpha_1)\\(a_1+\alpha-1,\alpha_1),(a_1+\beta-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right] \quad .. \quad (2.3)$$

where

$$p>n\geqslant 0,\ q\geqslant m\geqslant 2,\ \mathrm{Re}\,(1-c+\beta)\neq 0,-1,-2,\ldots,-(k-1).$$

$$\sum_{r=0}^{k}(-1)^{k-r}k_{c_r}\frac{\Gamma(\beta)}{\Gamma(\beta-r)}H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1-k+r,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+c-k-1,\alpha_1)\\(a_1+\alpha-k-1,\alpha_1),(a_1+\beta-k-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+c-k-1,\alpha_1)\\(a_1+\alpha-k-1,\alpha_1),(a_1+\beta-1,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right] \quad .. \quad (2.4)$$

where

$$p>n\geqslant 1,\ q\geqslant m\geqslant 2,\ \mathrm{Re}\,(a_1-k-\alpha_1\sigma)\neq 0,-1-2,\ldots,-(k-1).$$

$$\sum_{r=0}^{k} k_{c_r}(-1)^{-r}\frac{\Gamma(1-c)\,\Gamma(e-a+k-c-r)\,\Gamma(1+a+c-e+r-k)}{\Gamma(1-c-r)\,\Gamma(e+k-a)\,\Gamma(1+a-e-k)}$$

$$\times H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+a+c-e+r-k,\alpha_1)\\(a_1+a-1,\alpha_1),(a_1+a-e-k,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$=\frac{\Gamma(e-c-a)\,\Gamma(1+a+c-e)}{\Gamma(e-a)\,\Gamma(1+a-e)}H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+a-e+c,\alpha_1)\\(a_1+a-1,\alpha_1),(a_1+a-e,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right.\right] \quad .. \quad (2.5)$$

where

$$p > n \geqslant 0,\ q \geqslant m \geqslant 2,\ \mathrm{Re}\,(a_1+a+c-e-k-\alpha_1\sigma) \neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k} (-1)^{k-r} {}^{k}c_r \frac{\Gamma(c+k-r)\,\Gamma(e-a-c-k)\,\Gamma(1+a-e+c+k)}{\Gamma(c)\,\Gamma(e-a-r)\,\Gamma(1+a-e+r)}$$
$$\times H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+a+c+k-e,\alpha_1)\\(a_1+a-1,\alpha_1),(a_1+a-e+r,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]$$
$$=\frac{\Gamma(e-a-c)\,\Gamma(1+a+c-e)}{\Gamma(e-a)\,\Gamma(1+a-e)} H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+a+c-e,\alpha_1)\\(a_1+a-1,\alpha_1),(a_1+a-e,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right] \quad \ldots (2.6)$$

where

$$p > n \geqslant 0,\ q \geqslant m \geqslant 2,\ \mathrm{Re}\,(1-c-k) \neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k} (-1)^{k-r} {}^{k}c_r \frac{\Gamma(c-k)\,\Gamma(e-a-c+k-r)\,\Gamma(1+a-e+c+r-k)}{\Gamma(c)\,\Gamma(e-a-r)\,\Gamma(1+a-e+r)}$$
$$\times H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+a+c-e+r-k,\alpha_1)\\(a_1+a-1,\alpha_1),(a_1+a-e+r,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right]$$
$$=\frac{\Gamma(e-a-c)\,\Gamma(1-e+a+c)}{\Gamma(e-a)\,\Gamma(1+a-e)} H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(a_1+a+c-e,\alpha_1)\\(a_1+a-1,\alpha_1),(a_1+a-e,\alpha_1),(b_3,\beta_3),\ldots,(b_q,\beta_q)\end{matrix}\right] \quad \ldots (2.7)$$

where

$$p > n \geqslant 0,\ q \geqslant m \geqslant 2,\ \mathrm{Re}\,(a_1+a+c-e-k-\alpha_1\sigma) \neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

Proof of (2.1)

Expressing the H-function on the left of (2.1) in terms of Mellin-Barnes integral by virtue of (1.1) we get

$$\sum_{r=0}^{k} {}^{k}c_r(1+a_1-b_q)_r \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=3}^{n}\Gamma(1-a_j+\alpha_j s)\,\Gamma\left(1-a_1+\frac{k}{2}-r+\alpha_1 s\right)\Gamma\left(2+a_1+\frac{k}{2}+\alpha_1 s\right)}{\prod_{j=m+1}^{q-1}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\,\Gamma\left(2-b_q+\frac{k}{2}+\alpha_1 s\right)}$$
$$\times x^s\,ds.$$

On changing the order of integration and summation in the above series and using a well-known result (Erdélyi 1953, p. 3, (3)), we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=3}^{n}\Gamma(1-a_j+\alpha_j s)\,\Gamma\left(2+a_1+\frac{k}{2}+\alpha_1 s\right)\Gamma\left(1-a_1+\alpha_1 s+\frac{k}{2}\right)}{\prod_{j=m+1}^{q-1}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\,\Gamma\left(2-b_q+\frac{k}{2}+\alpha_1 s\right)}x^s$$
$$\times {}_2F_1\left[\begin{matrix}-k,\ 1+a_1-b_q\\ a_1-\frac{k}{2}-\alpha_1 s\end{matrix};\ 1\right]ds$$

where the contour L runs from $\sigma-i\infty$ to $\sigma+i\infty$ such that poles of $\prod_{j=1}^{m} \Gamma(b_j-\beta_j s)$ lie to the right of L and the poles of

$$\Gamma\left(2+a_1+\frac{k}{2}+\alpha_1 s\right),\ \Gamma\left(1-a_1+\frac{k}{2}+\alpha_1 s\right) \text{ and } \prod_{j=3}^{n} \Gamma(1-a_j+\alpha_j s)$$

lie to the left of L.

On expressing the Gauss's hypergeometric function involved in the above series in terms of Gamma-functions and interpreting the result thus obtained by virtue of (1.1), we get the required result after a little simplification. The proof is not complete unless we point out the conditions under which we are justified in expressing

$${}_2F_1\left[\begin{matrix} -k,\ 1+a_1-b_q \\ a_1-\frac{k}{2}-\alpha_1 s \end{matrix}\ ;\ 1\right] \quad (k \text{ is a non-negative integer})$$

in terms of Gamma-functions. In our case the condition merely comes out to be that $\text{Re}\left(a_1-\frac{k}{2}-\alpha\sigma\right) \neq 0,\ -1,\ -2, \ldots,\ -(k-1)$. The second condition which is a convergence condition is not necessary here since the hypergeometric function given above consists of only finite number of terms if expanded in a series.

The expansions (2.2) to (2.7) can be proved by proceeding on similar lines.

3. Particular Cases

Putting $\alpha_i = \beta_j = 1$ for $i = 1, \ldots, p$; $j = 1, \ldots, q$ in (2.1) to (2.7), we obtain the following expansions for the G-functions:

$$\sum_{r=0}^{k} k_{c_r}(1+a_1-b_q)_r G_{p,q}^{m,n}\left[z \left| \begin{matrix} a_1-\frac{k}{2}+r,\ -a_1-\frac{k}{2}-1,\ a_3, \ldots, a_p \\ b_1, \ldots, b_{q-2},\ b_{q-1},\ b_q-\frac{k}{2}-1 \end{matrix}\right.\right]$$
$$= G_{p,q}^{m,n}\left[z \left| \begin{matrix} a_1+\frac{k}{2},\ -a_1-\frac{k}{2}-1,\ a_3, \ldots, a_p \\ b_1, \ldots, b_{q-1},\ b_q+\frac{k}{2}-1 \end{matrix}\right.\right] \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where

$$p > n > 2,\ q > 1,\ \text{Re}\left(a_1-\frac{k}{2}-\sigma\right) \neq 0,\ -1,\ -2, \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k} k_{c_r}(1-\alpha)_r G_{p,q}^{m,n}\left[z \left| \begin{matrix} a_1, \ldots, a_{p-1},\ a_1+\beta-1 \\ a_1+\alpha+\beta-r-2,\ b_2, \ldots, b_q \end{matrix}\right.\right]$$
$$= G_{p,q}^{m,n}\left[z \left| \begin{matrix} a_1, \ldots, a_{p-1},\ a_1+\beta-k-1 \\ a_1+\alpha+\beta-k-2,\ b_2, \ldots, b_q \end{matrix}\right.\right] \quad \ldots \quad \ldots \quad (3.2)$$

where

$$p>n\geqslant 0,\ q\geqslant m\geqslant 1,\ \text{Re}\,(3-a_1-\alpha-\beta-\sigma)\neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k} {}^{k}c_r\,\Gamma(c-\beta-r)G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+c-1\\ a_1+\alpha-1,\,a_1+\beta+r-1,\,b_3,\ldots,b_q\end{matrix}\right]$$

$$=\Gamma(c-\beta-k)G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+c-k-1\\ a_1+\alpha-1,\,a_1+\beta-1,\,b_3,\ldots,b_q\end{matrix}\right] \quad \ldots \quad (3.3)$$

where

$$p>n\geqslant 0,\ q\geqslant m\geqslant 2,\ \text{Re}\,(1-c+\beta)\neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k}(-1)^{k-r}\,{}^{k}c_r\frac{\Gamma(\beta)}{\Gamma(\beta-r)}G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1+r-k,\,a_2,\ldots,a_{p-1},\,a_1+c-k-1\\ a_1+\alpha-k-1,\,a_1+\beta-k-1,\,b_3,\ldots,b_q\end{matrix}\right]$$

$$=G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+c-k-1\\ a_1+\alpha-k-1,\,a_1+\beta-1,\,b_3,\ldots,b_q\end{matrix}\right] \quad \ldots \quad \ldots \quad \ldots (3.4)$$

where

$$p>n\geqslant 1,\ q\geqslant m\geqslant 2,\ \text{Re}\,(a_1-k-\sigma)\neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k}(-1)^{-r}\,{}^{k}c_r\frac{\Gamma(1-c)\,\Gamma(e-a+k-c-r)\,\Gamma(1+a+c-e+r-k)}{\Gamma(1-c-r)\,\Gamma(e+k-a)\,\Gamma(1+a-e-k)}$$

$$\times G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+a+c-e+r-k\\ a_1+a-1,\,a_1+a-e-k,\,b_3,\ldots,b_q\end{matrix}\right]$$

$$=\frac{\Gamma(e-c-a)\,\Gamma(1+a+c-e)}{\Gamma(e-a)\,\Gamma(1+a-e)}G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+a-e+c\\ a_1+a-1,\,a_1+a-e,\,b_3,\ldots,b_q\end{matrix}\right] \quad (3.5)$$

where

$$p>n\geqslant 0,\ q\geqslant m\geqslant 2,\ \text{Re}\,(a_1+a+c-e-k-\sigma)\neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k}(-1)^{k-r}\,{}^{k}c_r\frac{\Gamma(c+k-r)\,\Gamma(e-a-c-k)\,\Gamma(1+a-e+c+k)}{\Gamma(c)\,\Gamma(e-a-r)\,\Gamma(1+a-e+r)}$$

$$\times G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+a+c+k-e\\ a_1+a-1,\,a_1+a-e+r,\,b_3,\ldots,b_q\end{matrix}\right]$$

$$=\frac{\Gamma(e-a-c)\,\Gamma(1+a+c-e)}{\Gamma(e-a)\,\Gamma(1+a-e)}G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+a+c-e\\ a_1+a-1,\,a_1+a-e,\,b_3,\ldots,b_q\end{matrix}\right] \quad (3.6)$$

where

$$p>n\geqslant 0,\ q\geqslant m\geqslant 2,\ \text{Re}\,(1-c-k)\neq 0,\ -1,\ -2,\ \ldots,\ -(k-1).$$

$$\sum_{r=0}^{k}(-1)^{k-r}\,{}^{k}c_r\frac{\Gamma(c-k)\,\Gamma(e-a-c+k-r)\,\Gamma(1+a-e+c+r-k)}{\Gamma(c)\,\Gamma(e-a-r)\,\Gamma(1+a-e+r)}$$

$$\times G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+a+c-e+r-k\\ a_1+a-1,\,a_1+a-e+r,\,b_3,\ldots,b_q\end{matrix}\right]$$

$$=\frac{\Gamma(e-a-c)\,\Gamma(1-e+a+c)}{\Gamma(e-a)\,\Gamma(1+a-e)}G_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},\,a_1+a+c-e\\ a_1+a-1,\,a_1+a-e,\,b_3,\ldots,b_q\end{matrix}\right] \quad (3.7)$$

where

$$p > n \geqslant 0,\ q \geqslant m \geqslant 2,\ \text{Re}\,(a_1+a+c-e-k-\sigma) \neq 0, -1, -2, \ldots, -(k-1).$$

If we take $m=1$, $n=p=q=2$ in (3.1); $n=1$, $m=q=p=2$ in (3.2) and $n=1$, $m=q=p=3$ in (3.3) to (3.7), then using known result [(Erdélyi 1953, p. 208(5); p. 209(g) in some cases)], we obtain the following expansions for the hypergeometric function, under conditions easily obtainable from (3.1) to (3.7).

$$\sum_{r=0}^{k} k_{c_r}(1+a_1-b_2)_r \frac{\Gamma\left(1+b_1-a_1+\frac{k}{2}-r\right)}{\Gamma\left(2+b_1-b_2+\frac{k}{2}\right)}\, {}_2F_1\left[\begin{matrix}1+b_1-a_1+\frac{k}{2}-r,\ 2+b_1+a_1+\frac{k}{2} \\ 2+b_1-b_2+\frac{k}{2}\end{matrix};x\right]$$

$$=\frac{\Gamma\left(1+b_1-a_1-\frac{k}{2}\right)}{\Gamma\left(2+b_1-b_2-\frac{k}{2}\right)}\, {}_2F_1\left[\begin{matrix}1+b_1-a_1-\frac{k}{2},\ 2+b_1+a_1+\frac{k}{2} \\ 2+b_1-b_2-\frac{k}{2}\end{matrix};x\right] \quad \ldots \quad \ldots \quad (3.8)$$

$$\sum_{r=0}^{k} k_{c_r}(1-\alpha)_r \frac{\Gamma(\alpha+\beta-r-1)}{\Gamma(\beta)}\, {}_2F_1\left[\begin{matrix}\alpha+\beta-r-1,\ 1+b_1-a_2 \\ \beta\end{matrix};x\right]$$

$$=\frac{\Gamma(\alpha+\beta-k-1)}{\Gamma(\beta-k)}\, {}_2F_1\left[\begin{matrix}\alpha+\beta-k-1,\ 1+b_1-a_2 \\ \beta-k\end{matrix};x\right] \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

$$\sum_{r=0}^{k} k_{c_r} \frac{\Gamma(b_2-a_2-r)\,\Gamma(a_2+r)}{\Gamma(b_2)}\, {}_3F_2\left[\begin{matrix}a_1,\ a_2+r,\ a_3 \\ b_1,\ b_2\end{matrix};x\right]$$

$$=\frac{\Gamma(b_2-a_2-k)\,\Gamma(a_2)}{\Gamma(b_2-k)}\, {}_3F_2\left[\begin{matrix}a_1,\ a_2,\ a_3 \\ b_1,\ b_2-k\end{matrix};x\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

$$\sum_{r=0}^{k} (-1)^{k-r} k_{c_r} \frac{\Gamma(\alpha-r)\,\Gamma(1+b_2+k-r-a_1)x^{k-r}}{\Gamma(1+a_2-a_1+k-r)\,\Gamma(c-r)}\, {}_3F_2\left[\begin{matrix}\alpha-r,\ \beta-r,\ 1+b_2-a_1+k-r \\ c-r,\ 1+a_2-a_1+k-r\end{matrix};-x\right]$$

$$=\frac{\Gamma(\alpha-k)\,\Gamma(1+b_2-a_1)}{\Gamma(1+a_2-a_1)\,\Gamma(c-k)}\, {}_3F_2\left[\begin{matrix}\alpha-k,\ \beta,\ 1+b_2-a_1 \\ c-k,\ 1+a_2-a_1\end{matrix};-x\right] \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

$$\sum_{r=0}^{k} (-1)^{-r} k_{c_r} \frac{\Gamma(1-c)\,\Gamma(e-a-c+k-r)}{\Gamma(1-c-r)\,\Gamma(e+k-a)}\, {}_3F_2\left[\begin{matrix}a,\ 1+a-e-k,\ 1+b_2-a_1 \\ 1+a+c-e+r-k,\ 1+a_2-a_1\end{matrix};x\right]$$

$$=\frac{\Gamma(e-c-a)}{\Gamma(e-a)}\, {}_3F_2\left[\begin{matrix}a,\ 1+a-e,\ 1+b_2-a_1 \\ 1+a+c-e,\ 1+a_2-a_1\end{matrix};x\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.12)$$

$$\sum_{r=0}^{k} (-1)^{k-r} k_{c_r} \frac{\Gamma(c+k-r)\,\Gamma(e-a-c-k)}{\Gamma(c)\,\Gamma(e-r-a)}\, {}_3F_2\left[\begin{matrix}a,\ 1+a-e+r,\ 1+b_2-a_1 \\ 1+a-e+c+k,\ 1+a_2-a_1\end{matrix};x\right]$$

$$=\frac{\Gamma(e-a-c)}{\Gamma(e-a)}\, {}_3F_2\left[\begin{matrix}a,\ 1+a-e,\ 1+b_2-a_1 \\ 1+a+c-e,\ 1+a_2-a_1\end{matrix};x\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.13)$$

$$\sum_{r=0}^{k} (-1)^{k-r} k_{c_r} \frac{\Gamma(c-k)\,\Gamma(e-a-c+k-r)}{\Gamma(c)\,\Gamma(e-a-r)} {}_3F_2\left[\begin{matrix} a,\, 1+a-e+r,\, 1+b_2-a_1 \\ 1+r-e+a+c-k,\, 1+a_2-a_1 \end{matrix}; x\right]$$

$$= \frac{\Gamma(e-a-c)}{\Gamma(e-a)} {}_3F_2\left[\begin{matrix} a,\, 1+a-e,\, 1+b_2-a_1 \\ 1+a+c-e,\, 1+a_2-a_1 \end{matrix}; x\right]. \quad .. \quad .. \quad .. \quad .. \; (3.14)$$

If we put $a_1 = b_1$ in (3.10), $a_2 = b_2$ in (3.11) and take the confluent form of ${}_2F_1$ in (3.9), we obtain the expansions given recently by Pathan (1969, 2.2.9, 2.2.15, 2.3.10).

Also when $k = 1$ in (3.10) to (3.14) and $a_1 = b_1$ in (3.10), $a_2 = b_2$ in (3.11) to (3.14), we get the results given by Bose (1950, p. 202) and Kulshreshtha (1967, p. 13).

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