# THE PROBABILITIES FOR SEVERAL CONSECUTIVE EIGENVALUES OF A RANDOM MATRIX

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We study statistical properties of the eigenvalues of a random matrix belonging to the so-called Gaussian unitary and Gaussian orthogonal ensembles. An expression is given for the probability that an interval of length $2t$ contains exactly $n$ levels (their positions are specified or not). This result is used to derive the probability density of the spacing between a level and its $n$th neighbour (the positions of the $(n-1)$ intermediate levels are specified or not).

## 1. Introduction

To explain the statistical properties of energy levels of complex systems one usually studies, as a model, the eigenvalues of a random matrix (Porter 1965). The random matrices can be chosen from a great variety of ensembles (Porter 1965). Here we shall be concerned only with the so-called Gaussian ensembles defined as follows.

(i) *Gaussian orthogonal ensemble*—The matrices are real symmetric. All the matrix elements $M_{ij}$ with $i \geq j$ are independent random variables. The joint probability density $P(M)$ for $M$ is invariant under all real orthogonal transformations.

(ii) *Gaussian unitary ensemble*—The matrices are hermitian. The real and imaginary parts of $M_{ij}$ with $i > j$ and the diagonal elements $M_{ii}$ are independent random variables. The joint probability density $P(M)$ for $M$ is invariant under all unitary transformations.

It can be shown (Mehta 1967, Chapter 3) that the joint probability density for the eigenvalues of such matrices is given by

$$P_{N\beta}(x_1, \ldots, x_N) = C_{N,\beta} \cdot \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \cdot \exp\left(-\tfrac{1}{2}\beta \sum_{j=1}^{N} x_j^2\right) \tag{1.1}$$

where $N$ is the order of the matrices and $\beta$ is a parameter; $\beta = 1$ for the orthogonal ensemble and $\beta = 2$ for the unitary ensemble.*

* The value $\beta = 4$ corresponds to the Gaussian symplectic ensemble, where the matrix elements of $M$ are real quaternions, $M$ is self-dual, the various parts of the elements $M_{ij}$, $i \geq j$ are independent random variables and $P(M)$ is invariant under all symplectic transformations. The statistical properties of the eigenvalues of such matrices are identical to those of an alternate series from the orthogonal ensemble. The later can be treated in a straightforward manner.

The constant $C_{N\beta}$ is

$$C_{N\beta} = (2\pi)^{-N/2}\beta^{\frac{1}{2}N+\frac{1}{4}\beta N(N-1)}\{\Gamma(1+\tfrac{1}{2}\beta)\}^N \Big/ \prod_{j=1}^{N} \Gamma(1+\tfrac{1}{2}\beta j). \quad \ldots \quad (1.2)$$

It is chosen so that

$$\int_{-\infty}^{\infty} \cdots \int P_{N\beta}(x_1, \ldots, x_N)\, dx_1 \ldots dx_N = 1.$$

It is of interest to keep in expression (1.1) a few variables, say $n$ in number, fixed at particular values near the origin and to integrate over all the other variables. If these other variables are integrated from $-\infty$ to $\infty$ without any restriction we get the $n$-level correlation function (Mehta 1971). On the other hand, we may integrate over these variables from $-\infty$ to $-\theta$ and from $\theta$ to $\infty$, thus leaving empty an interval of length $2\theta$ which contains all the non-integrated variables around the origin. This gives us the probability that an interval of length $2\theta$ contains exactly $n$ levels situated at positions $x_1, \ldots, x_n$, namely:

$$A_\beta(\theta; x_1, \ldots, x_n) = \frac{N!}{n!\,(N-n)!} \int_{(\text{out})} \cdots \int P_{N\beta}(x_1, \ldots, x_N)\, dx_{n+1} \ldots dx_N. \quad (1.3)$$

The quantities of interest are those which remain finite in the limit $N \to \infty$. We put

$$\left.\begin{aligned} t &= \frac{1}{\pi}(2N)^{\frac{1}{2}}\theta \\ y_j &= \frac{1}{\pi}(2N)^{\frac{1}{2}}x_j, \quad j = 1, \ldots, n \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

and define

$$B_\beta(t; y_1, \ldots, y_n) = \lim_{N\to\infty} (\pi/(2N)^{\frac{1}{2}})^n A_\beta(\theta; x_1, \ldots, x_n). \quad \ldots \quad (1.5)$$

This is the probability that in a series of eigenvalues with mean spacing unity an interval of length $2t$ contains exactly $n$ levels at positions $y_1, \ldots, y_n$. To get the probability that a randomly chosen interval of length $S = 2t$ contains exactly $n$ levels one has to integrate $B_\beta(t; y_1, \ldots, y_n)$ over $y_1, \ldots, y_n$ from $-t$ to $t$,

$$E_\beta(n; S) = \int_{-S/2}^{S/2} \cdots \int B_\beta(S/2; y_1, \ldots, y_n)\, dy_1 \ldots dy_n. \quad \ldots \quad (1.6)$$

If we put $y_1 = -t$, $y_2 = t$ and integrate over $y_3, \ldots, y_n$ from $-t$ to $t$, we get the $(n-2)$th neighbour spacing distribution

$$p_\beta(n-2; S) = n(n-1) \int_{-S/2}^{S/2} \cdots \int B_\beta\left(\frac{S}{2}; -\frac{S}{2}, \frac{S}{2}, y_3, \ldots, y_n\right) dy_3 \ldots dy_n. \quad (1.7)$$

The $p_\beta(n; S)$ and $E_\beta(n; S)$ are of course related (Mehta 1967, Appendix A.11),

$$\frac{d^2}{dS^2}E_\beta(0; S) = p_\beta(0; S), \quad \frac{d^2}{dS^2}E_\beta(1; S) = p_\beta(1; S) - 2p_\beta(0; S)$$

$$\frac{d^2}{dS^2}E_\beta(n; S) = p_\beta(n; S) - 2p_\beta(n-1; S) + p_\beta(n-2; S), \quad n \geq 2 \quad \ldots \quad (1.8)$$

or

$$p_\beta(n; S) = \sum_{j=0}^{n} (n-j+1) \frac{d^2}{dS^2} E_\beta(j; S). \quad \ldots \quad (1.9)$$

The $E_\beta(n; S)$ satisfy the relation

$$\sum_{n=0}^{\infty} E_\beta(n; S) = 1. \quad \ldots \quad (1.10)$$

To get the probability that $n-1$ consecutive spacings have values $S_1, \ldots, S_{n-1}$ it is sufficient first to order the $y_j$ as

$$-t \leq y_1 \leq \ldots \leq y_n \leq t \quad \ldots \quad (1.11)$$

and then to substitute

$$t = -y_1 = \tfrac{1}{2} \sum_{j=1}^{n-1} s_j \quad \ldots \quad (1.12)$$

$$y_i = \sum_{j=1}^{i-1} s_j - \tfrac{1}{2} \sum_{j=1}^{n-1} s_j, \quad i = 2, \ldots, n \quad \ldots \quad (1.13)$$

$$(\text{thus } y_n = t)$$

in the expression for $B_\beta(t; y_1, \ldots, y_n)$.

Explicit expressions are derived for $B_\beta(t; y_1, \ldots, y_n)$, $E_\beta(n; S)$ and $p_\beta(n; S)$ for the cases $\beta = 1$ and $\beta = 2$. For the case $\beta = 4$ two alternative expressions for $E_4(n; S)$ are given in Appendix C. Formulae for $B_4$ and $p_4$ are easy to derive and cumbersome to write; we will simply omit them.

## 2. The Unitary Ensemble: Case $\beta = 2$

To calculate the integral (1.3) it is convenient to introduce the harmonic oscillator functions

$$\phi_j(x) = (2^j j!\sqrt{\pi})^{-\frac{1}{2}} \cdot e^{x^2/2}\left(-\frac{d}{dx}\right)^j e^{-x^2} \quad \ldots \quad (2.1)$$

and write $P_{N2}(x_1, \ldots, x_N)$ as a determinant (Mehta 1967, Chapter 6.1) with elements $\sum_k \phi_i(x_k)\phi_j(x_k)$:

$$\left.\begin{aligned} P_{N2}(x_1, \ldots, x_N) &= \frac{1}{N!}\{\det[\phi_{i-1}(x_j)]_{i,j=1,\ldots,N}\}^2 \\ &= \frac{1}{N!}\det\left[\sum_{k=1}^{N} \phi_i(x_k)\phi_j(x_k)\right]_{i,j=0,\ldots,N-1} \end{aligned}\right\}. \quad (2.2)$$

We want to calculate

$$A_2(\theta; x_1, \ldots, x_n) = \frac{N!}{n!\,(N-n)!}\int_{(\text{out})} \cdots \int P_{N2}(x_1, \ldots, x_N)\, dx_{n+1} \ldots dx_N. \quad (2.3)$$

The subscript 'out' means that the variables $x_1, \ldots, x_N$ belong to the intervals

$$\left.\begin{aligned} |x_j| &\leq \theta, \quad \text{if } 1 \leq j \leq n \\ |x_j| &\geq \theta, \quad \text{if } n+1 \leq j \leq N \end{aligned}\right\}. \quad \ldots \quad (2.4)$$

In eqn. (2.2) the row with index $i$ can be considered as a sum of $N$ rows with elements $\phi_i(x_k)\phi_j(x_k)$, $k = 1, \ldots, N$. Expressing each row as such sums, we write the determinant in eqn. (2.2) as a sum of determinants. If a variable occurs in two or more rows, these rows are proportional and therefore the corresponding determinant is zero. The non-zero determinants are those in which all the indices $k$ are different.

$$P_{N2}(x_1, \ldots, x_N) = \frac{1}{N!}\sum_{(k_i)} \det\,[\phi_i(x_{k_i})\phi_j(x_{k_i})]_{i,\,j=0,\ldots,N-1}\,. \qquad (2.5)$$

The indices $k_0, \ldots, k_{N-1}$ are obtained by permutation of $1, \ldots, N$ and in eqn. (2.5) the summation is over all permutations. As each variable $x_1, \ldots, x_N$ occurs in one and only one row of any of the determinants of eqn. (2.5) one may easily integrate over as many variables as one likes. After integrating over $x_{n+1}, \ldots, x_N$ we expand each determinant in the Laplace manner according to the $n$ rows containing the variables $x_1, \ldots, x_n$. Introducing the matrix $g$ with elements

$$g_{ij} = \delta_{ij} - \int_{-\theta}^{\theta} \phi_i(x)\phi_j(x)\,dx \qquad (2.6)$$

we obtain the result:

$$A_2(\theta; x_1, \ldots, x_n) = \frac{1}{n!}\sum_{(i,\,j)} \{\det\,[\phi_{i_k}(x_k)\phi_{j_l}(x_k)]_{k,\,l=1,\ldots,n}\}\, G_{i_1,\ldots,i_n;\,j_1,\ldots,j_n}\,. \qquad (2.7)$$

The indices $i_1, \ldots, i_n$ are chosen from $1, \ldots, N$ as also the indices $j_1, \ldots, j_n$. The indices $i_1, \ldots, i_n$ are not ordered, while the indices $j_1, \ldots, j_n$ are ordered $j_1 < \ldots < j_n$. The summation in (2.7) is extended over all possible choices of indices satisfying the above conditions. The cofactor $G_{i_1,\ldots,i_n;\,j_1,\ldots,j_n}$ is apart from a sign the determinant of the $(N-n)\times(N-n)$ matrix obtained from $g$ by omitting the rows $i_1, \ldots, i_n$ and the columns $j_1, \ldots, j_n$. The sign is plus or minus according as $\sum_{k=1}^{n} (i_k+j_k)$ is even or odd. Therefore (Gantmacher 1959)

$$G_{i_1,\ldots,i_n;\,j_1,\ldots,j_n} = \det\,[g]\,.\,\det\left\{[(g^{-1})_{ji}]_{\substack{j=j_1,\ldots,j_n\\ i=i_1,\ldots,j_n}}\right\}. \qquad (2.8)$$

Let us briefly recall (Mehta 1967, Chapters 6.1, 5.5) the diagonalization of $g$. For the case $n = 0$, eqn. (2.7) reads

$$\begin{aligned} A_2(\theta) &= \det\,[g] \\ &= \det\left[\delta_{ij} - \int_{-\theta}^{\theta} \phi_i(x)\phi_j(x)\,dx\right]_{i,\,j=0,\ldots,N-1} \\ &= \prod_{i=0}^{N-1} (1-\lambda_i) \end{aligned} \qquad (2.9)$$

where the $\lambda_i$, $i = 0, 1, \ldots, N-1$, are the eigenvalues of the matrix $[\gamma]$ with elements

$$\gamma_{ij} = \int_{-\theta}^{\theta} \phi_i(x)\phi_j(x)\,dx \qquad (2.10)$$

i.e.

$$\sum_{j=0}^{N-1} \gamma_{ij} h_{jk} = h_{ik}\lambda_k. \quad \text{(2.11)}$$

Equivalently, the $\lambda_i$ are the eigenvalues of the integral equation

$$\lambda_i \psi_i(x) = \int_{-\theta}^{\theta} K(x, y)\psi_i(y)\, dy \quad \text{(2.12)}$$

with the kernel

$$K(x, y) = \sum_{i=0}^{N-1} \phi_i(x)\phi_i(y) \quad \text{(2.13)}$$

and the eigenfunctions

$$\psi_i(x) = \sum_{j=0}^{N-1} h_{ji}\phi_j(x). \quad \text{(2.14)}$$

The normalization of the $\psi_i(x)$ depends on that of the eigenvectors of $\gamma$. As $\gamma$ is real and symmetric we may choose (Mehta 1967, Appendix A.23.5) its eigenvectors to be real and normalized to unity, so that $h = [h_{ij}]$ is a real orthogonal matrix:

$$\sum_{j=0}^{N-1} h_{ij}h_{kj} = \sum_{j=0}^{N-1} h_{ji}h_{jk} = \delta_{ik}. \quad \text{(2.15)}$$

From eqns. (2.13)–(2.15) we deduce

$$K(x, y) = \sum_{i=0}^{N-1} \psi_i(x)\psi_i(y). \quad \text{(2.16)}$$

Since $K$ is real and symmetric, its eigenfunctions $\psi_i(x)$ are orthogonal

$$\int_{-\theta}^{\theta} \psi_i(x)\psi_j(x)\, dx = 0, \quad i \neq j. \quad \text{(2.17)}$$

However, their normalization is not unity. From eqns. (2.12), (2.16) and (2.17) one gets

$$\int_{-\theta}^{\theta} \psi_i^2(x)\, dx = \lambda_i. \quad \text{(2.18)}$$

To take the limits as $N \to \infty$, let us put

$$D = \pi/(2N)^{\frac{1}{2}}, \quad \theta = Dt, \quad x = Dt\xi, \quad y = Dt\eta \quad \text{(2.19)}$$

and keep $\xi$, $\eta$, $t$ finite. Then it is known that (Mehta 1967, Appendix A.9)

$$K(x, y) \underset{N\to\infty}{\simeq} (Dt)^{-1} Q(\xi, \eta) \quad \text{(2.20)}$$

$$\psi_i(x) \underset{N\to\infty}{\simeq} a_i f_i(\xi) \quad \text{(2.21)}$$

where

$$Q(\xi, \eta) = \frac{\sin (\xi-\eta)\pi t}{(\xi-\eta)\pi} \quad \text{(2.22)}$$

and where the $f_i(\xi)$ are the spheroidal functions, depending on the parameter $t$, solutions of the integral equation

$$\lambda_i f_i(\xi) = \int_{-1}^{1} Q(\xi, \eta) f_i(\eta)\, d\eta. \quad \text{(2.23)}$$

If we normalize the spheroidal functions as

$$\int_{-1}^{1} f_i(\xi) f_j(\xi)\, d\xi = \delta_{ij} \quad \ldots \quad (2.24)$$

then the constants $a_i$ are given by

$$\lambda_i = \int_{-\theta}^{\theta} \psi_i^2(x)\, dx = a_i^2 Dt \int_{-1}^{1} f_i^2(\xi)\, d\xi$$

or

$$a_i = (\lambda_i / Dt)^{\frac{1}{2}}. \quad \ldots \quad (2.25)$$

After this short digression, let us come back to eqns. (2.7) and (2.8). The matrix $h$ diagonalizes $g$ and hence also $g^{-1}$, while it transforms the functions $\phi_i(x)$ into $\psi_i(x)$. Using eqns. (2.11), (2.14) and (2.15) we can write eqns. (2.7) and (2.8) as

$$A_2(\theta;\, x_1, \ldots, x_n) = \frac{1}{n!} \sum_{(i,j)} \det \left[\psi_{i_k}(x_k)\psi_{j_l}(x_k)\right]_{k,\, l=1, \ldots, n} . \det\, [1-\Lambda] . \det \left[(1-\Lambda)^{-1}_{ji}\right]_{\substack{j=j_1, \ldots, j_n \\ i=i_1, \ldots, i_n}} \quad \ldots \quad (2.26)$$

where $\Lambda$ is the diagonal matrix with diagonal elements $\lambda_i$. As $[1-\Lambda]$ is diagonal, non-vanishing terms result if and only if the indices $i$ can be obtained by a permutation $P$ of the indices $j$:

$$j_1 < \ldots < j_n, \quad i_k = j_{Pk}, \quad k = 1, \ldots, n. \quad \ldots \quad (2.27)$$

Thus eqn. (2.26) can be written as

$$A_2(\theta;\, x_1, \ldots, x_n) = \frac{1}{n!} \prod_{\rho=0}^{N-1} (1-\lambda_\rho) . \sum_{(j)} \frac{1}{1-\lambda_{j_1}} \cdots \frac{1}{1-\lambda_{j_n}}$$

$$. \sum_{P} (-1)^P \det \left[\psi_{j_{Pk}}(x_k)\psi_{j_l}(x_k)\right]_{k,\, l=1, \ldots, n} \quad \ldots \quad (2.28)$$

which coincides with

$$A_2(\theta;\, x_1, \ldots, x_n) = \frac{1}{n!} \prod_{\rho=0}^{N-1} (1-\lambda_\rho) . \sum_{(j)} \frac{1}{1-\lambda_{j_1}} \cdots \frac{1}{1-\lambda_{j_n}} \left\{\det \left[\psi_{j_l}(x_k)\right]_{k,\, l=1, \ldots, n}\right\}^2. \quad \ldots \quad (2.29)$$

When $N \to \infty$, $\theta \to 0$ while $t$ and $y_j$ given by equation (1.4) are finite, we get from equations (1.4), (1.5), (2.29) and (2.19), (2.21):

$$B_2(t;\, y_1, \ldots, y_n) = \frac{t^{-n}}{n!} \prod_{\rho} (1-\lambda_\rho) \sum_{(j)} \frac{\lambda_{j_1}}{1-\lambda_{j_1}} \cdots \frac{\lambda_{j_n}}{1-\lambda_{j_n}} \left(\det \left\{\left[f_{j_l}(y_k/t)\right]_{l,\, k=1, \ldots, n}\right\}\right)^2. \quad \ldots \quad (2.30)$$

Let us remind that the eigenvalues $\lambda_i$ and the functions $f_i(\xi)$ depend on $t$ as a parameter and that the indices $0 \leqslant j_1 < \ldots < j_n$ are integers.

The preceding formula can also be written in the following form:

$$B_2(t;\, y_1, \ldots, y_n) = \frac{t^{-n}}{n!} \prod_{\rho} (1-\lambda_\rho) \sum_{(j)} \frac{\lambda_{j_1}}{1-\lambda_{j_1}} \cdots \frac{\lambda_{j_n}}{1-\lambda_{j_n}} \det \left\{\left[\sum_{k=1}^{n} f_{j_l}(y_k/t) f_{j_m}(y_k/t)\right]_{l,\, m=1, \ldots, n}\right\}. \quad \ldots \quad (2.31)$$

In this equation, we may integrate immediately over $y_1, \ldots, y_n$. Using the orthonormality condition of the $f_i(\xi)$ [*see* eqn. (2.24)], we obtain from (1.6) and (2.31):

$$E_2(n;\ S) = \prod_{\rho} (1-\lambda_\rho) \sum_{j_1 < \ldots < j_n} \frac{\lambda_{j_1}}{1-\lambda_{j_1}} \cdots \frac{\lambda_{j_n}}{1-\lambda_{j_n}}. \quad \ldots \quad \ldots (2.32)$$

In the same way, from (1.7), (2.30), (2.31) and (2.24), we get

$$p_2(n-2;\ S) = \frac{4}{S^2} \prod_{\rho} (1-\lambda_\rho) \sum_{(j)} \frac{\lambda_{j_1}}{1-\lambda_{j_1}} \cdots \frac{\lambda_{j_n}}{1-\lambda_{j_n}}$$

$$\cdot \sum_{l,\, m=1}^{n} f_{j_l}(1) f_{j_m}(-1) [f_{j_l}(1) f_{j_m}(-1) - f_{j_l}(-1) f_{j_m}(1)]. \quad \ldots \quad \ldots (2.33)$$

On the other hand, to obtain the probability density of $(n-1)$ consecutive spacings $S_1, \ldots, S_{n-1}$ one has only to make the substitutions (1.12), (1.13) in the expression (2.30) or (2.31) as explained at the end of section 1.

The case $n = 2$ is of special interest. Putting $S = 2t$, we get the probability density for a single spacing $S$ as:

$$p_2(0;\ S) = B_2(\tfrac{1}{2}S;\ -\tfrac{1}{2}S,\ \tfrac{1}{2}S) = \frac{16}{S^2} \prod_{\rho} (1-\lambda_\rho) \sum_{i} \frac{\lambda_{2i}}{1-\lambda_{2i}} f_{2i}^2(1) \sum_{j} \frac{\lambda_{2j+1}}{1-\lambda_{2j+1}} f_{2j+1}^2(1). \quad \ldots (2.34)$$

Using eqns. (1.8) and (2.32), we may also write

$$p_2(0;\ S) = \frac{d^2}{dS^2} E_2(0;\ S) = \frac{d^2}{dS^2} \prod_{\rho} (1-\lambda_\rho) \quad \ldots \quad \ldots (2.35)$$

a result previously obtained (Mehta 1967, Chapter 6.1).

However, for numerical calculations, the expression (2.34) is better as will now be explained. For $t = 0$ the spheroidal functions $f_i(\xi)$ are proportional to the Legendre polynomials

$$f_i(\xi) = \left(\frac{2i+1}{2}\right)^{\frac{1}{2}} P_i(\xi), \quad (t = 0)$$

and for $t < 1$ they may be expanded in terms of them. Extensive numerical tables (Stratton *et al.* 1956) are available for these expansion coefficients. Using them one can calculate the $\lambda_i$, $f_i(\xi)$ or any derivative of $f_i(\xi)$ to the same accuracy as the coefficients themselves (Mehta 1967, Chapter 5.5). If a function is known only numerically, its derivative is known with a lesser precision. Equation (2.35) involves two numerical differentiations, hence its precision is considerably less than that of (2.34) which involves no numerical differentiation.

## 3. The Orthogonal Ensemble: Case $\beta = 1$

Because of the absolute value in eqn. (1.1) this case requires more care. We would like to calculate

$$A_1(\theta; x_1, \ldots, x_n) = \frac{N!}{n!\,(N-n)!}\int_{(\text{out})}\cdots\int P_{N1}(x_1, \ldots, x_N)\, dx_{n+1}\ldots dx_N \quad (3.1)$$

where $|x_j| \leq \theta$, $j = 1, \ldots, n$ and the subscript 'out' means, as in section 2, that in integrating over $x_{n+1}, \ldots, x_N$ the interval between $-\theta$ and $\theta$ is excluded. We shall use the method of integration over alternate variables (Mehta 1967, Chapter 5.2). The calculation is considerably simplified if $N$ and $n$ are both even.

Let us order the variables $x_1 \leq \ldots \leq x_N$, and write $P_{N1}(x_1, \ldots, x_N)$ as a determinant whose elements are harmonic oscillator functions (2.1) and their derivatives (Mehta 1967, Chapter 5.2). We take $N$ even, $N = 2m$.

$$P_{2m,1}(x_1, \ldots, x_{2m}) = C_{2m,1}\,.\prod_{0}^{2m-1}(2^{-j}j!\,\sqrt{\pi})^{\frac{1}{2}}\,.\det[\phi_{i-1}(x_j)]_{i,j=1,\ldots,2m}. \quad (3.2)$$

Using the relation

$$(2j+1)^{\frac{1}{2}}\phi_{2j+1}(x) = -\sqrt{2}\,\phi'_{2j}(x) + (2j)^{\frac{1}{2}}\phi_{2j-1}(x), \quad \ldots \quad (3.3)$$

one has

$$\det[\phi_{i-1}(x_j)]_{i,j=1,\ldots,2m} = (-1)^m \prod_{j=1}^{m}\left(\frac{2}{2j-1}\right)^{\frac{1}{2}} \det\begin{bmatrix}\phi_{2i}(x_j)\\ \phi'_{2i}(x_j)\end{bmatrix}_{\substack{i=0,\ldots,m-1\\ j=1,\ldots,2m.}} \quad (3.4)$$

Putting together eqns. (1.2), (3.2) and (3.4) we get

$$P_{2m,1}(x_1, \ldots, x_{2m}) = \frac{(-2)^{-m}}{(2m)!}\det\begin{bmatrix}\phi_{2i}(x_j)\\ \phi'_{2i}(x_j)\end{bmatrix}_{\substack{i=0,1,\ldots,m-1\\ j=1,2,\ldots,2m}} \quad \ldots \quad (3.5)$$

or written in full

$$P_{2m,1}(x_1, \ldots, x_{2m}) = \frac{(-2)^m}{(2m)!}\begin{vmatrix}\phi_0(x_1) & \phi_0(x_2)\ldots & \phi_0(x_{2m})\\ \phi'_0(x_1) & \phi'_0(x_2)\ldots & \phi'_0(x_{2m})\\ \cdots & \cdots & \cdots\\ \phi_{2m-2}(x_1) & \phi_{2m-2}(x_2)\ldots & \phi_{2m-2}(x_{2m})\\ \phi'_{2m-2}(x_1) & \phi'_{2m-2}(x_2)\ldots & \phi'_{2m-2}(x_{2m})\end{vmatrix}. \quad (3.6)$$

If $n = 0$, we get (Mehta 1967, Chapter 5.5)

$$A_1(\theta) = \int_{(\text{out})}\cdots\int P_{2m,1}(x_1, \ldots, x_{2m})\, dx_1\ldots dx_{2m} = \det[\bar{g}_{ij}]_{i,j=0,\ldots,m-1} \quad \ldots \quad (3.7)$$

where the matrix $\bar{g}$ has the elements

$$\bar{g}_{ij} = \delta_{ij} - \int_{-\theta}^{\theta}\phi_{2i}(x)\,\phi_{2j}(x)\,dx. \quad \ldots \quad (3.8)$$

The diagonalization of $\bar{g}$ and the passage to the limit $m \to \infty$, $\theta \to 0$ proceeds exactly as in section 2, except that we now need only the even part in $x$ of

$K(x, y)$, and only the even functions $\psi_{2i}(x)$ and $f_{2i}(\xi)$. Equations corresponding to eqns. (2.9) to (2.25) are

$$A_1(\theta) = \prod_i (1-\lambda_{2i}) \qquad (3.9)$$

$$\bar{\gamma}_{ij} = \int_{-\theta}^{\theta} \phi_{2i}(x)\, \phi_{2j}(x)\, dx \qquad (3.10)$$

$$\sum_{j=0}^{m-1} \bar{\gamma}_{ij} \bar{h}_{jk} = \bar{h}_{ik}\, \lambda_{2k} \qquad (3.11)$$

$$\lambda_{2i}\, \psi_{2i}(x) = \int_{-\theta}^{\theta} \bar{K}(x, y)\, \psi_{2i}(y)\, dy \qquad (3.12)$$

$$\bar{K}(x, y) = \sum_{i=0}^{m-1} \phi_{2i}(x)\, \phi_{2i}(y) = \sum_{j=0}^{m-1} \psi_{2j}(x)\, \psi_{2j}(y) \qquad (3.13)$$

$$\psi_{2i}(x) = \sum_{j=0}^{m-1} \bar{h}_{ji} \phi_{2j}(x) \qquad (3.14)$$

$$\sum_{j=0}^{m-1} \bar{h}_{ij} \bar{h}_{kj} = \sum_{j=0}^{m-1} \bar{h}_{ji} \bar{h}_{jk} = \delta_{ik} \qquad (3.15)$$

$$D = \pi/(2\sqrt{m}), \quad \theta = Dt, \quad x = Dt\xi, \quad y = Dt\eta \qquad (3.16)$$

$$\bar{K}(x, y) \underset{m\to\infty}{\simeq} (Dt)^{-1} \bar{Q}(\xi, \eta), \qquad (3.17)$$

$$\bar{Q}(\xi, \eta) = \tfrac{1}{2}\{Q(\xi, \eta)+Q(\xi, -\eta)\}. \qquad (3.18)$$

The symbols $Q(\xi, \eta)$, $\psi_i(x)$, $f_i(\xi)$, $\lambda_i$ and $a_i$ have the same meaning as in section 2. Equation (2.25) may be completed by

$$\psi_{2i}^{(j)}(\xi) \underset{m\to\infty}{\simeq} (Dt)^{-j-\frac{1}{2}} \lambda_{2i}^{\frac{1}{2}} f_{2i}^{(j)}(\xi) \qquad (3.19)$$

where the superscript $j$ denotes the $j$th derivative.

Let us now come back to eqns. (3.1) and (3.6). In writing the expression (3.6) we took the ordering of the variables as $x_1 \leq \ldots \leq x_{2m}$. However, the same expressions is valid also when $-\theta \leq x_1 \leq \ldots \leq x_{2r} \leq \theta$, $x_{2r+1} \leq x_{2r+2} \leq \ldots \leq x_{2m}$, and $|x_j| \leq \theta$ for $j = 2r+1, 2r+2, \ldots, 2m$. This pertains to the fact that the determinant in eqn. (3.6) does not change sign when the columns containing the variables $x_{2r+1}, \ldots, x_{2m}$ are passed one by one over the $2r$ columns containing the variables $x_1, \ldots, x_{2r}$. Let us therefore take $n = 2r$, $r \geq 1$. We expand the determinant in eqn. (3.6) by the first $2r$ columns in the Laplace manner and integrate every term so obtained over $x_{2r+1}, \ldots, x_{2m}$ outside the interval $(-\theta, \theta)$, while $-\theta \leq x_1 \leq \ldots \leq x_{2r} \leq \theta$, using the method of integration over alternate variables (Mehta 1967, Chapter 5.2). Let us remark that the $2r \times 2r$ determinants formed from the first $2r$ columns of (3.6) not all will have a non-zero coefficient. Only those containing

$r$ rows of functions $\phi$ and another $r$ rows of functions $\phi'$ will survive. This is so because

$$\int_{(\text{out})\, y \le x} \int \{\phi_{2i}(y)\phi_{2j}(x) - \phi_{2i}(x)\phi_{2j}(y)\}\, dx\, dy = \int_{(\text{out})\, y \le x} \int \{\phi'_{2i}(y)\phi'_{2j}(x) - \phi'_{2i}(x)\phi_{2j}(y)\}\, dx\, dy = 0. \quad .. \ (3.20)$$

We also have

$$\int_{(\text{out})\, y \le x} \int \{\phi_{2i}(y)\phi'_{2j}(x) - \phi_{2i}(x)\phi'_{2j}(y)\}\, dx\, dy = 2\bar{g}_{ij}. \quad .. \ (3.21)$$

The result is

$$A_1(\theta; x_1, \ldots, x_{2r}) = \frac{(-2)^{-r}}{(2r)!} \sum_{(i,j)} \det \left\{ \begin{bmatrix} \phi_{2i_k}(x_l) \\ \phi'_{2j_k}(x_l) \end{bmatrix}_{\substack{k=1,\ldots,r \\ l=1,\ldots,2r}} \right\} . \bar{G}_{i_1,\ldots,i_r;\, j_1,\ldots,j_r} \quad .. \ (3.22)$$

where the summation is extended over all possible choices of the indices $i_1 < \ldots < i_r$, $j_1 < \ldots < j_r$ from $0, \ldots, m-1$. The $G_{i_1,\ldots,i_r;\, j_1,\ldots,j_r}$ is apart from a sign the $(m-r)\times(m-r)$ determinant obtained from $\bar{g}$ by omitting the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_r$. The sign is plus or minus according as $\sum_{k=1}^{r} (i_k + j_k)$ is even or odd. Thus $G_{i_1,\ldots,i_r;\, j_1,\ldots,j_r}$ is equal to (Gantmacher 1959) the determinant of $\bar{g}$ multiplied by an $r\times r$ determinant formed from the elements of the inverse of $\bar{g}$

$$\bar{G}_{i_1,\ldots,i_r;\, j_1,\ldots,j_r} = \det\,[g] \,.\, \det \left\{ [(g^{-1})_{ji}]_{\substack{j=j_1,\ldots,j_r \\ i=i_1,\ldots,i_r}} \right\}. \quad .. \ (3.23)$$

Diagonalizing $\bar{g}$ as explained above, and using eqns. (3.8) to (3.15), we get

$$A_1(\theta; x_1, \ldots, x_{2r}) = \frac{(-2)^{-r}}{(2r)!} \prod_{l=0}^{m-1} (1-\lambda_{2l}) \,.\, \sum_{(i)} (1-\lambda_{2i_1})^{-1} \ldots (1-\lambda_{2i_r})^{-1} \,.\, \det \left\{ \begin{bmatrix} \psi_{2i_k}(x_j) \\ \psi'_{2i_k}(x_j) \end{bmatrix}_{\substack{k=1,\ldots,r \\ j=1,\ldots,2r}} \right\}. \quad .. \ (3.24)$$

With

$$D = \pi/(2\sqrt{m}), \quad \theta = Dt, \quad x_j = Dy_j \quad .. \quad .. \quad .. \ (3.25)$$

we get in the limit $m \to \infty$

$$B_1(t; y_1, \ldots, y_{2r}) = \lim D^{2r} A_1(Dt; Dy_1, \ldots, Dy_{2r}) = \frac{(-2)^{-r}}{(2r)!} t^{-2r} \prod_{l} (1-\lambda_{2l})$$

$$\cdot \sum_{(i)} \frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_r}}{1-\lambda_{2i_r}} \,.\, \det \left\{ \begin{bmatrix} f_{2i_k}(y_j/t) \\ f'_{2i_k}(y_j/t) \end{bmatrix}_{\substack{k=1,\ldots,r \\ j=1,\ldots,2r}} \right\}. \quad (3.26)$$

The variables $y_j$ satisfy of course the inequalities

$$-t \le y_1 \le \ldots \le y_{2r} \le t \quad .. \quad .. \quad .. \ (3.27)$$

the $\lambda_{2i}$ and the $f_{2i}(\xi)$ depend on $t$ as a parameter and the indices $i_1 < \ldots < i_r$ are non-negative integers.

To get $E_1(2r; S)$, eqn. (1.6), from (3.26) one may use again the method of integration over alternate variables (Mehta 1967, Chapter 5.2). It is easy to see that

$$\int_{-1 \le y \le x \le 1}\int dy\, dx\, \{f_{2i}(y) f_{2j}(x) - f_{2i}(x) f_{2j}(y)\} = \int_{-1 \le y \le x \le 1}\int dy\, dx\, \{f'_{2i}(y) f'_{2j}(x) - f'_{2i}(x) f'_{2j}(y)\} = 0 \quad \ldots \quad (3.28)$$

and that

$$\int_{-1 \le y \le x \le 1}\int dy\, dx\, \{f_{2i}(y) f'_{2j}(x) - f_{2i}(x) f'_{2j}(y)\} = -2\Big(\delta_{ij} - f_{2j}(1)\int_{-1}^{1} f_{2i}(y)\, dy\Big). \quad \ldots \quad (3.29)$$

Therefore, the method of integration over alternate variables gives us

$$E_1(2r; S) = \prod_l (1-\lambda_{2l}) \,.\, \sum_{i_1 < \ldots < i_r} \frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_r}}{1-\lambda_{2i_r}} \,.\, \det\Big[\delta_{ij} - f_{2i}(1)\int_{-1}^{1} f_{2j}(y)\, dy\Big]_{i,j=i_1, i_2, \ldots, i_r} \quad (3.30)$$

or (*see* Appendix A)

$$E_1(2r; S) = \prod_l (1-\lambda_{2l}) \sum_{i_1 < \ldots < i_r} \left\{\frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_r}}{1-\lambda_{2i_r}} \,.\, \left(1 - \sum_{j=1}^{r} f_{2i_j}(1) \int_{-1}^{1} f_{2i_j}(y)\, dy\right)\right\}. \quad \ldots \quad (3.31)$$

As we obtained (3.22) by integrating the expression (3.4) so we obtain from (3.26) the following expression for $p_1(2r-2; S)$

$$p_1(2r-2; S) = -\frac{2}{S^2}\prod_l (1-\lambda_{2l}) \,.\, \sum_{i_1 < \ldots < i_r} \left\{\frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{1-\lambda_{2i_r}}{\lambda_{2i_r}} \,.\, \sum_{j,k} a_{jk} b_{jk}\right\} \quad (3.32)$$

where the summation over $j$, $k$ in the above equation is over the indices $i_1, \ldots, i_r$, while

$$a_{jk} = \det\begin{bmatrix} f_{2j}(-1) & f_{2j}(1) \\ f'_{2k}(-1) & f'_{2k}(1) \end{bmatrix} = 2 f_{2j}(1) f'_{2k}(1) \quad \ldots \quad \ldots \quad (3.33)$$

and $b_{jk}$ is the cofactor of the element $(j, k)$ in

$$\det\Big[\delta_{ij} - f_{2j}(1)\int_{-1}^{1} f_{2i}(y)\, dy\Big]_{i,j=i_1,\ldots,i_r}. \quad \ldots \quad \ldots \quad (3.34)$$

That is to say (cf. Appendix A)

$$b_{jk} = f_{2j}(1)\int_{-1}^{1} f_{2k}(y)\, dy + \delta_{jk}\Big\{1 - \sum_l f_{2l}(1)\int_{-1}^{1} f_{2l}((y)\, dy\Big\} \quad \ldots \quad (3.35)$$

where the index $l$ takes all the values $i_1, \ldots, i_r$.

We took $n$ even, $n = 2r$, for convenience. If $n$ is odd, $n = 2r-1$, one can proceed as in chapter 16 of Mehta (1967). We shall omit the lengthy details and content ourselves with the final formulae

$$B_1(t, y_1, \ldots, y_{2r-1}) = \frac{(-2)^{-r+1}}{(2r-1)!} t^{-2r+1} \prod_l (1-\lambda_{2l})$$
$$\cdot \sum_{(i)} \frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_r}}{1-\lambda_{2i_r}} f_{2j}(1) \det M_j(i_1, \ldots, i_r) \quad \ldots \quad \ldots \quad (3.36)$$

where the $(2r-1)\times(2r-1)$ matrix $M_j(i_1, \ldots, i_r)$ depends on the variables $y_1, \ldots, y_{2r-1}$ as:

$$M_j(i_1, \ldots, i_r) = \begin{bmatrix} f_{2j}\left(\frac{y_1}{t}\right) & f_{2i_1}\left(\frac{y_1}{t}\right) & f'_{2i_1}\left(\frac{y_1}{t}\right) \cdots & f_{2i_r-1}\left(\frac{y_1}{t}\right) & f'_{2i_r-1}\left(\frac{y_1}{t}\right) \\ f_{2j}\left(\frac{y_{2r-1}}{t}\right) & f_{2i_1}\left(\frac{y_{2r-1}}{t}\right) & f'_{2i_1}\left(\frac{y_{2r-1}}{t}\right) \cdots & f_{2i_r-1}\left(\frac{y_{2r-1}}{t}\right) & f'_{2i_r-1}\left(\frac{y_{2r-1}}{t}\right) \end{bmatrix}$$
$$\ldots \quad (3.37)$$

$j$ is one of the indices $i_1, \ldots, i_r$ and the summation in eqn. (3.36) is over all possible choices of non-negative integers $0 \leqslant i_1 < \ldots < i_r$ and of $j$ among these integers. The variables $y_j$ are supposed to be ordered $-t \leqslant y_1 \leqslant \ldots \leqslant y_{2r-1} \leqslant t$. Integration over the $y_j$ from $-t$ to $t$ gives

$$E_1(2r-1;\ S = \prod_l (1-\lambda_{2l}) \cdot \sum_{(i)} \frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_r}}{1-\lambda_{2i_r}}$$
$$\cdot \left\{ \sum_{j=1}^{r} f_{2i_j}(1) \int_{-1}^{1} f_{2i_j}(y)\, dy \right\} . \quad \ldots \quad \ldots \quad (3.38)$$

The expression for $p_1(2r-1;\ S)$ is obtained by putting $y_1 = -t$, $y_{2r-1} = t$ and integrating over the other variables. We get

$$p_1(2r-3;\ S) = -\frac{2}{S^2} \prod_l (1-\lambda_{2l}) \cdot \sum_{(i)} \frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_r}}{1-\lambda_{2i_r}}$$
$$\sum_{j, j_1, j_2, j_3} a_{j_1 j_2 j_3} b_{j_2 j_3;\, j_1 j} \cdot f_{2j}(1). \quad \ldots \quad (3.39)$$

By definition

$$a_{j_1 j_2 j_3} = 2f'_{2j_1}(1) \left\{ f_{2j_2}(1) \int_{-1}^{1} f_{2j_3}(y)\, dy - f_{2j_3}(1) \int_{-1}^{1} f_{2j_2}(y)\, dy \right\} \quad \ldots \quad (3.40)$$

and $b_{j_2 j_3;\, j_1 j}$ is, apart from a sign, the determinant obtained from (3.34) by omitting the rows corresponding to the values $j_2$, $j_3$ and the columns corresponding to the values of $j_1$, $j$. The sign can be fixed, as always, by bringing this minor to the leading position in the upper left-hand corner. The indices $j, j_1, j_2, j_3$ are chosen from $i_1, \ldots, i_r$ and the summation in eqn. (3.39) is over all such choices and then over all choices of the non-negative integers $0 \leqslant i_1 < \ldots < i_r$.

The special case $r = 1$ is of particular interest. It gives the probability density for a single spacing $S = 2t$ as

$$p_1(0;\, S) = B_1(\tfrac{1}{2}S;\, -\tfrac{1}{2}S,\, \tfrac{1}{2}S) = -\frac{2}{S^2}(1-\lambda_{2i})\,.\sum_i \frac{\lambda_{2i}}{1-\lambda_{2i}} \det\begin{bmatrix} f_{2i}(-1) & f'_{2i}(-1) \\ f_{2i}(1) & f'_{2i}(1) \end{bmatrix}$$

$$= -\frac{4}{S^2}\prod_i(1-\lambda_{2i})\,.\sum_i \frac{\lambda_{2i}}{1-\lambda_{2i}} f_{2i}(1)f'_{2i}(1). \quad .. \ (3.41)$$

The probability $p_1(0;\, S)$ can also be calculated by using eqn. (1.8)

$$p_1(0;\, S) = \frac{d^2}{dS^2}E_1(0;\, S) = \frac{d^2}{dS^2}\prod_i(1-\lambda_{2i}) \quad .. \quad .. \ (3.42)$$

a known result (Mehta 1967, Chapter 5.5). However, if the spheroidal functions $f_{2i}(\xi)$ are known in terms of Legendre polynomials with a certain numerical accuracy of the expansion coefficients, then, as we noted at the end of section 2, the numbers obtained by using eqn. (3.41) are considerably more precise than those obtained from (3.42).

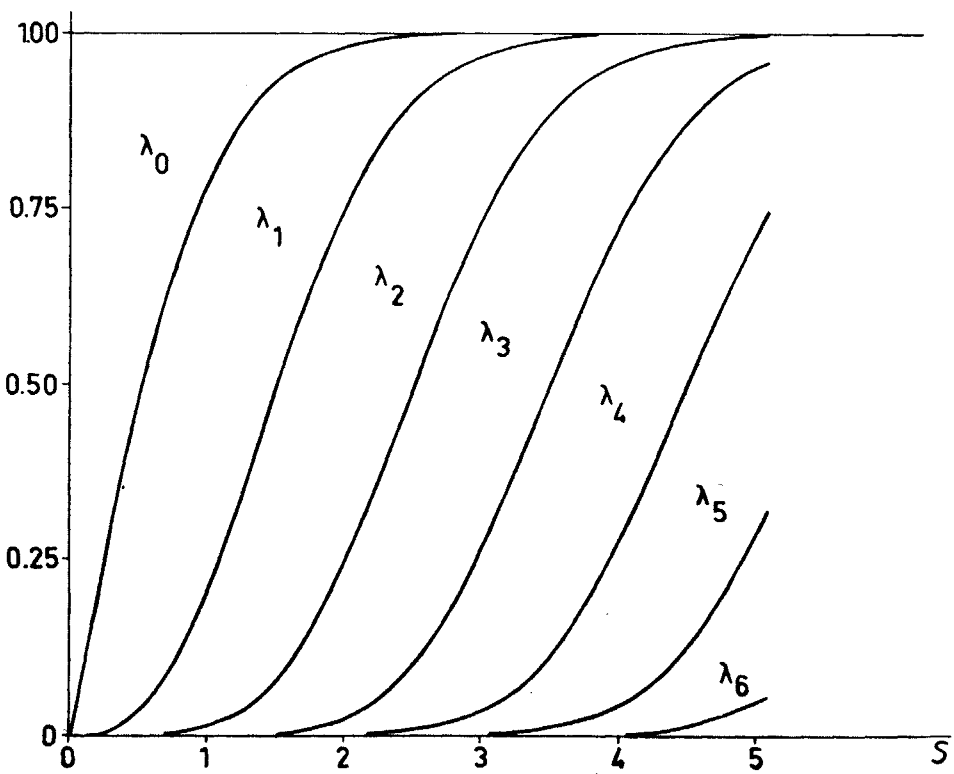


FIG. 1. The eigenvalues $\lambda_i$ as a function of $S$.

## 4. Numerical Results and Power Series Expansions

For $t < 1$ the spheroidal functions may be expanded (Stratton *et al.* 1956, Robin 1959) in terms of Legendre polynomials,

$$f_i(x) = \sum_j{}' d_j(i, t) P_j(x) \quad \ldots \quad (4.1)$$

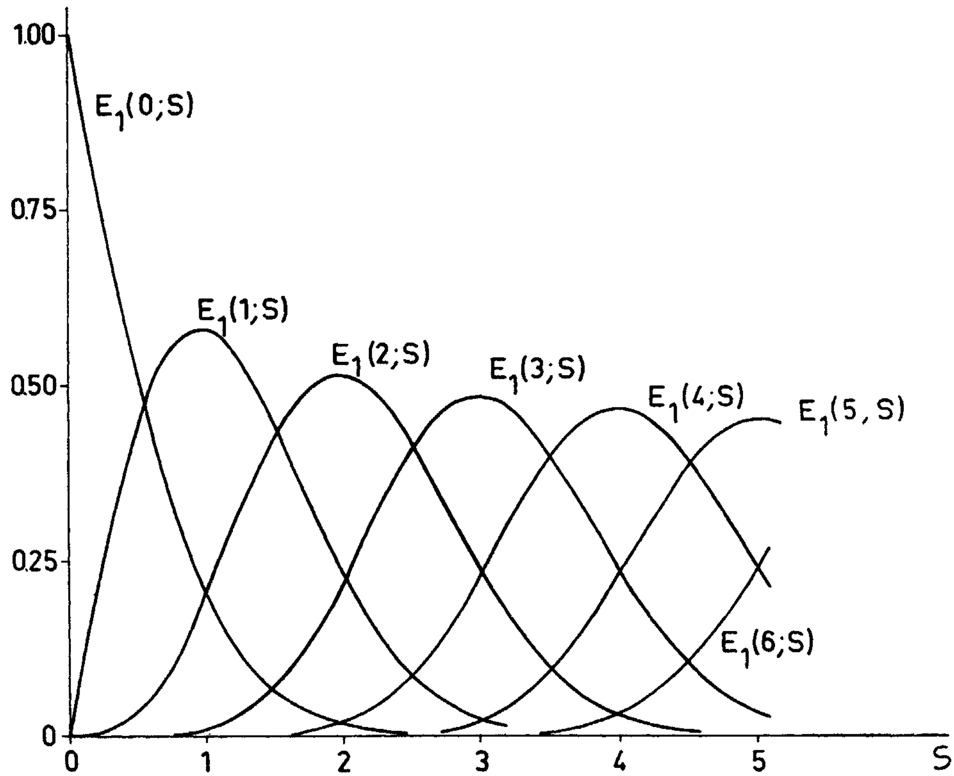


Fig. 2. The $E_1(n; S)$ for $n = 0, 1, 2, 3, 4, 5, 6$.

where the prime on the summation sign indicates that only those $j$ occur for which $(j-i)$ is an even integer. Extensive numerical tables for the coefficients $d_j(i, t)$ are available (Stratton *et al.* 1956). To calculate the $\lambda_i$ it is convenient to note that the kernel $Q(\xi, \eta)$ in eqn. (2.22) is the square of the kernel

$$q(\xi, \eta) = (\tfrac{1}{2}t)^{\frac{1}{2}} \cdot e^{i\xi\eta\pi t} \quad \ldots \quad (4.2)$$

i.e. that

$$\int_{-1}^{1} q(\xi, \eta) \cdot q^*(\eta, \zeta)\, d\eta = Q(\xi, \zeta). \quad \ldots \quad (4.3)$$

Let us denote by $\mu_i$ the eigenvalues of the integral equation

$$\mu_i f_i(\xi) = \int_{-1}^{1} q(\xi, \eta) f_i(\eta)\, d\eta. \quad \ldots \quad (4.4)$$

The eigenfunctions $f_i(\xi)$ are either even or odd and may be chosen to be real. Thus, they are also eigenfunctions of the complex conjugate integral equation. Then eqn. (4.3) implies that

$$\lambda_i = \mu_i \,.\, \mu_i^* = |\mu_i|^2. \qquad (4.5)$$

Setting $\xi = 0$ in eqn. (4.4) we get

$$\mu_i f_i(0) = (\tfrac{1}{2}t)^{\frac{1}{2}} \int_{-1}^{1} f_i(\eta)\, d\eta \qquad (4.6)$$

while by differentiating (4.4) once with respect to $\xi$ and by setting $\xi = 0$ we get

$$\mu_i f_i'(0) = (\tfrac{1}{2}t)^{\frac{1}{2}} \,.\, i\pi t \int_{-1}^{1} \eta f_i(\eta)\, d\eta. \qquad (4.7)$$

Equations (4.5) and (4.6) give

$$\lambda_{2i} = \tfrac{1}{2}t \left\{ \int_{-1}^{1} f_{2i}(\eta)\, d\eta / f_{2i}(0) \right\}^2 \qquad (4.8)$$

while eqns. (4.5) and (4.7) give

$$\lambda_{2i+1} = \frac{\pi^2}{2} t^3 \left\{ \int_{-1}^{1} \eta f_{2i+1}(\eta)\, d\eta / f_{2i+1}'(0) \right\}^2 \qquad (4.9)$$

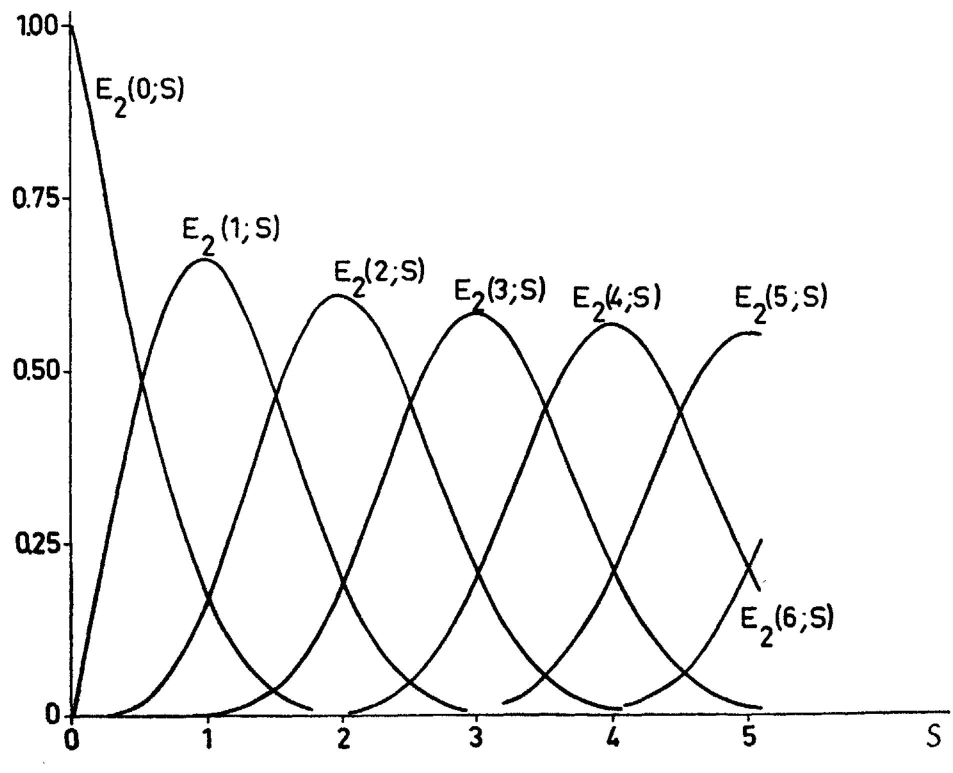


FIG. 3. The $E_2(n;\, S)$ for $n = 0, 1, 2, 3, 4, 5, 6$.

where $i$ is a non-negative integer. Equations (4.1), (4.8) and (4.9) then give (Gaudin 1961 or Mehta 1967, Chapter 5.5)

$$\lambda_{2i} = 2t\left\{d_0(2i,\, t)\bigg/\sum_j (-1)^j \frac{1\,.\,3\ldots(2j-1)}{2\,.\,4\ldots(2j)}\, d_{2j}(2i,\, t)\right\}^2 \quad \ldots \quad (4.10)$$

and (Kahn 1963 or Mehta 1967, Chapter 6.1)

$$\lambda_{2i+1} = \tfrac{2}{9}\,\pi^2 t^3 \left\{d_1(2i+1,\, t)\bigg/\sum_j (-1)^j \frac{3\,.\,5\ldots(2j+1)}{2\,.\,4\ldots(2j)}\, d_{2j+1}(2i+1,\, t)\right\}^2. \quad (4.11)$$

Thus from a knowledge of the coefficients $d_j(i,\, t)$ one may calculate the eigenvalues $\lambda_i$. The spheroidal functions $f_i(x)$ are of course given by eqn. (4.1).

We give in Appendix B a few terms of the power-series expansions of $\lambda_i$, $f_i(x)$, $E_\beta(n;\, S)$ and $p_\beta(n;\, S)$ for low values of $i$ and $n$ of interest. Tables I to III give $\lambda_i$, $0 \leq i \leq 8$, $E_1(n;\, S)$, $0 \leq n \leq 7$, and $E_2(n;\, S)$, $0 \leq n \leq 7$ for $0 \leq S \lesssim 5$. Figs. 1 to 3 give a graphical representation of these functions. For comparison the corresponding probabilities

$$E_0(n\,;\, S) = \frac{1}{n\,!}\, S^n e^{-S} \quad \ldots \quad (4.12)$$

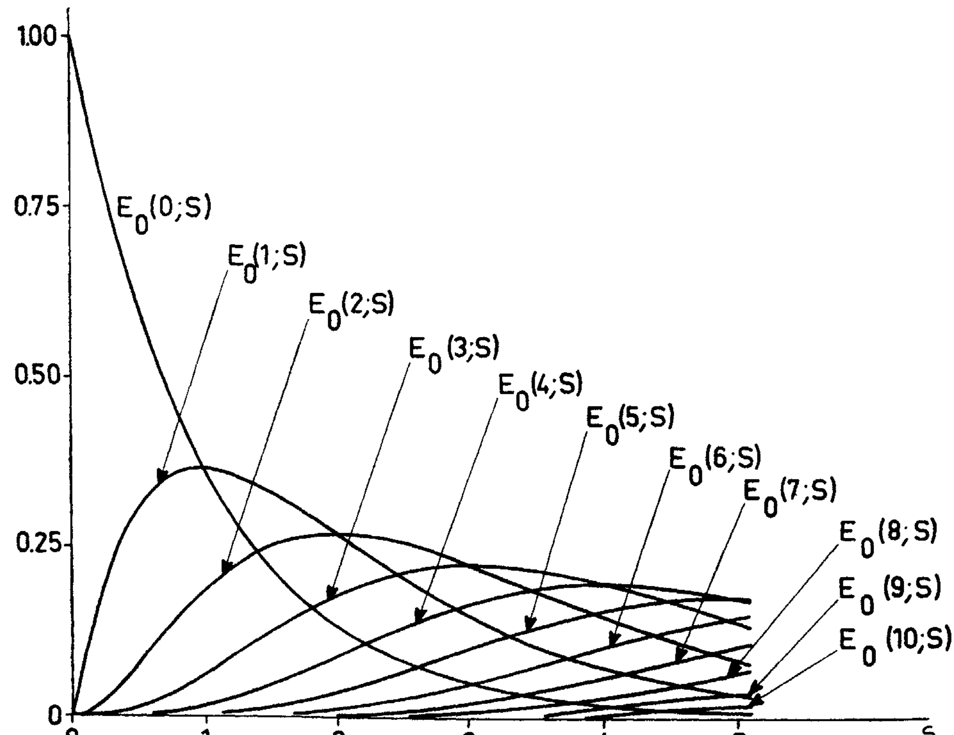


FIG. 4. The $E_0(n;\, S) = \dfrac{1}{n!}\, S^n e^{-s}$, for $0 \leq n \leq 10$ (Poisson probabilities).

TABLE I

*The eigenvalues $\lambda_i$ of the integral eqn.* (2.23) *for* $0 \leq i \leq 8$, $0 \leq S \leq 5$

| $S$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $\lambda_7$ | $\lambda_8$ |
|---|---|---|---|---|---|---|---|---|---|
| 0· | 0· | 0· | | | | | | | |
| 0·127 | 0·12676 | 0·00056 | | | | | | | |
| 0·255 | 0·25019 | 0·00444 | 0·00001 | | | | | | |
| 0·382 | 0·36724 | 0·01464 | 0·00010 | | | | | | |
| 0·509 | 0·47534 | 0·03355 | 0·00041 | | | | | | |
| 0·637 | 0·57258 | 0·06279 | 0·00124 | 0·00001 | | | | | |
| 0·891 | 0·73072 | 0·15395 | 0·00650 | 0·00010 | | | | | |
| 1·146 | 0·84107 | 0·28243 | 0·02186 | 0·00056 | 0·00001 | | | | |
| 1·401 | 0·91141 | 0·43134 | 0·05555 | 0·00223 | 0·00004 | | | | |
| 1·655 | 0·95288 | 0·57972 | 0·11544 | 0·00698 | 0·00020 | | | | |
| 1·910 | 0·97583 | 0·70996 | 0·20514 | 0·01820 | 0·00071 | 0·00002 | | | |
| 2·165 | 0·98793 | 0·81233 | 0·32102 | 0·04102 | 0·00214 | 0·00007 | | | |
| 2·419 | 0·99409 | 0·88538 | 0·45226 | 0·08154 | 0·00567 | 0·00022 | 0·00001 | | |
| 2·674 | 0·99715 | 0·93336 | 0·58414 | 0·14510 | 0·01339 | 0·00065 | 0·00002 | | |
| 2·928 | 0·99864 | 0·96278 | 0·70298 | 0·23361 | 0·02866 | 0·00172 | 0·00007 | | |
| 3·183 | 0·99936 | 0·97987 | 0·79992 | 0·34356 | 0·05602 | 0·00418 | 0·00019 | 0·00001 | |
| 3·438 | 0·99970 | 0·98937 | 0·87226 | 0·46601 | 0·10050 | 0·00937 | 0·00052 | 0·00002 | |
| 3·692 | 0·99986 | 0·99450 | 0·92218 | 0·58890 | 0·16613 | 0·01950 | 0·00128 | 0·00006 | |
| 3·947 | 0·99994 | 0·99719 | 0·95442 | 0·70071 | 0·25388 | 0·03781 | 0·00295 | 0·00015 | 0·00001 |
| 4·202 | 0·99997 | 0·99859 | 0·97415 | 0·79357 | 0·36019 | 0·06844 | 0·00639 | 0·00039 | 0·00002 |
| 4·456 | | 0·99930 | 0·98571 | 0·86457 | 0·47705 | 0·11572 | 0·01306 | 0·00091 | 0·00005 |
| 4·711 | | 0·99965 | 0·99226 | 0·91499 | 0·59392 | 0·18282 | 0·02520 | 0·00202 | 0·00012 |
| 4·966 | | 0·99983 | 0·99588 | 0·94863 | 0·70069 | 0·27007 | 0·04598 | 0·00429 | 0·00028 |
| 5·093 | | 0·99988 | 0·99701 | 0·96055 | 0·74790 | 0·32028 | 0·06078 | 0·00613 | 0·00042 |

TABLE II

*The probabilities $E_1(n; S)$ of having n eigenvalues of a real symmetric matrix in an interval S for $0 \leq n \leq 7$, $0 \leq S \leq 5$*

| $S$ | $E_1(0;S)$ | $E_1(1;S)$ | $E_1(2;S)$ | $E_1(3;S)$ | $E_1(4;S)$ | $E_1(5;S)$ | $E_1(6;S)$ | $E_1(7;S)$ |
|---|---|---|---|---|---|---|---|---|
| 0· | 1· | 0· | 0· | | | | | |
| 0·127 | 0·87324 | 0·12620 | 0·00056 | | | | | |
| 0·255 | 0·74980 | 0·24576 | 0·00444 | | | | | |
| 0·382 | 0·63270 | 0·35267 | 0·01460 | 0·00004 | | | | |
| 0·509 | 0·52445 | 0·44200 | 0·03336 | 0·00019 | | | | |
| 0·637 | 0·42689 | 0·51031 | 0·06209 | 0·00071 | | | | |
| 0·891 | 0·26753 | 0·57844 | 0·14928 | 0·00473 | 0·00002 | | | |
| 1·146 | 0·15546 | 0·56171 | 0·26444 | 0·01823 | 0·00016 | | | |
| 1·401 | 0·08367 | 0·48373 | 0·38194 | 0·04971 | 0·00096 | | | |
| 1·655 | 0·04167 | 0·37568 | 0·47249 | 0·10612 | 0·00402 | 0·00002 | | |
| 1·910 | 0·01920 | 0·26555 | 0·51452 | 0·18780 | 0·01279 | 0·00014 | | |
| 2·165 | 0·00818 | 0·17178 | 0·50144 | 0·28523 | 0·03269 | 0·00068 | | |
| 2·419 | 0·00322 | 0·10203 | 0·44206 | 0·38031 | 0·06982 | 0·00254 | 0·00002 | |
| 2·674 | 0·00117 | 0·05577 | 0·35500 | 0·45211 | 0·12814 | 0·00773 | 0·00009 | |
| 2·928 | 0·00039 | 0·02808 | 0·26096 | 0·48437 | 0·20603 | 0·01978 | 0·00039 | |
| 3·183 | 0·00012 | 0·01304 | 0·17617 | 0·47130 | 0·29444 | 0·04352 | 0·00140 | 0·00001 |
| 3·438 | 0·00004 | 0·00559 | 0·10946 | 0·41885 | 0·37802 | 0·08372 | 0·00429 | 0·00005 |
| 3·692 | 0·00001 | 0·00221 | 0·06270 | 0·34136 | 0·43954 | 0·14273 | 0·01126 | 0·00020 |
| 3·947 | | 0·00081 | 0·03315 | 0·25590 | 0·46572 | 0·21791 | 0·02581 | 0·00071 |
| 4·202 | | 0·00027 | 0·01618 | 0·17684 | 0·45179 | 0·30040 | 0·05228 | 0·00223 |
| 4·456 | | 0·00008 | 0·00730 | 0·11282 | 0·40273 | 0·37638 | 0·09453 | 0·00608 |
| 4·711 | | 0·00002 | 0·00304 | 0·06652 | 0·33077 | 0·43087 | 0·15345 | 0·01459 |
| 4·966 | | 0·00001 | 0·00117 | 0·03628 | 0·25083 | 0·45252 | 0·22692 | 0·03118 |
| 5·093 | | | 0·00071 | 0·02602 | 0·21209 | 0·44948 | 0·26607 | 0·04375 |

TABLE III

*The probabilities $E_2(n; S)$ of having n eigenvalues of a hermitian matrix in an interval S for $0 \leq n \leq 7$, $0 \leq S \leq 5$*

| $S$ | $E_2(0; S)$ | $E_2(1; S)$ | $E_2(2; S)$ | $E_2(3; S)$ | $E_2(4; S)$ | $E_2(5; S)$ | $E_2(6; S)$ | $E_2(7; S)$ |
|---|---|---|---|---|---|---|---|---|
| 0· | 1· | 0· | 0· | | | | | |
| 0·127 | 0·87275 | 0·12718 | 0·00007 | | | | | |
| 0·255 | 0·74647 | 0·25242 | 0·00111 | | | | | |
| 0·382 | 0·62344 | 0·37115 | 0·00541 | | | | | |
| 0·509 | 0·50685 | 0·47700 | 0·01614 | 0·00001 | | | | |
| 0·637 | 0·40008 | 0·56327 | 0·03661 | 0·00005 | | | | |
| 0·891 | 0·22632 | 0·65684 | 0·11610 | 0·00074 | | | | |
| 1·146 | 0·11149 | 0·63644 | 0·24674 | 0·00533 | | | | |
| 1·401 | 0·04747 | 0·52729 | 0·40249 | 0·02270 | 0·00005 | | | |
| 1·655 | 0·01739 | 0·37809 | 0·53689 | 0·06717 | 0·00046 | | | |
| 1·910 | 0·00547 | 0·23561 | 0·60522 | 0·15101 | 0·00270 | | | |
| 2·165 | 0·00147 | 0·12759 | 0·58709 | 0·27268 | 0·01114 | 0·00002 | | |
| 2·419 | 0·00034 | 0·05992 | 0·49506 | 0·40982 | 0·03468 | 0·00019 | | |
| 2·674 | 0·00007 | 0·02433 | 0·36473 | 0·52458 | 0·08518 | 0·00111 | | |
| 2·928 | 0·00001 | 0·00852 | 0·23518 | 0·58041 | 0·17104 | 0·00482 | 0·00001 | |
| 3·183 | | 0·00257 | 0·13262 | 0·56033 | 0·28814 | 0·01627 | 0·00007 | |
| 3·438 | | 0·00066 | 0·06528 | 0·47461 | 0·41494 | 0·04411 | 0·00040 | |
| 3·692 | | 0·00015 | 0·02797 | 0·35367 | 0·51765 | 0·09868 | 0·00188 | |
| 3·947 | | 0·00003 | 0·01040 | 0·23199 | 0·56466 | 0·18595 | 0·00696 | 0·00002 |
| 4·202 | | | 0·00335 | 0·13381 | 0·54181 | 0·29997 | 0·02093 | 0·00013 |
| 4·456 | | | 0·00093 | 0·06772 | 0·45897 | 0·41949 | 0·05222 | 0·00066 |
| 4·711 | | | 0·00022 | 0·02999 | 0·34379 | 0·51330 | 0·10998 | 0·00271 |
| 4·966 | | | 0·00005 | 0·01159 | 0·22769 | 0·55320 | 0·19836 | 0·00908 |
| 5·093 | | | 0·00002 | 0·00684 | 0·17688 | 0·54839 | 0·25230 | 0·01549 |

for a set of independent random levels are drawn in Fig. 4 and the probabilities $E_\infty(n;\, S)$ corresponding to equally spaced levels in Fig. 5.

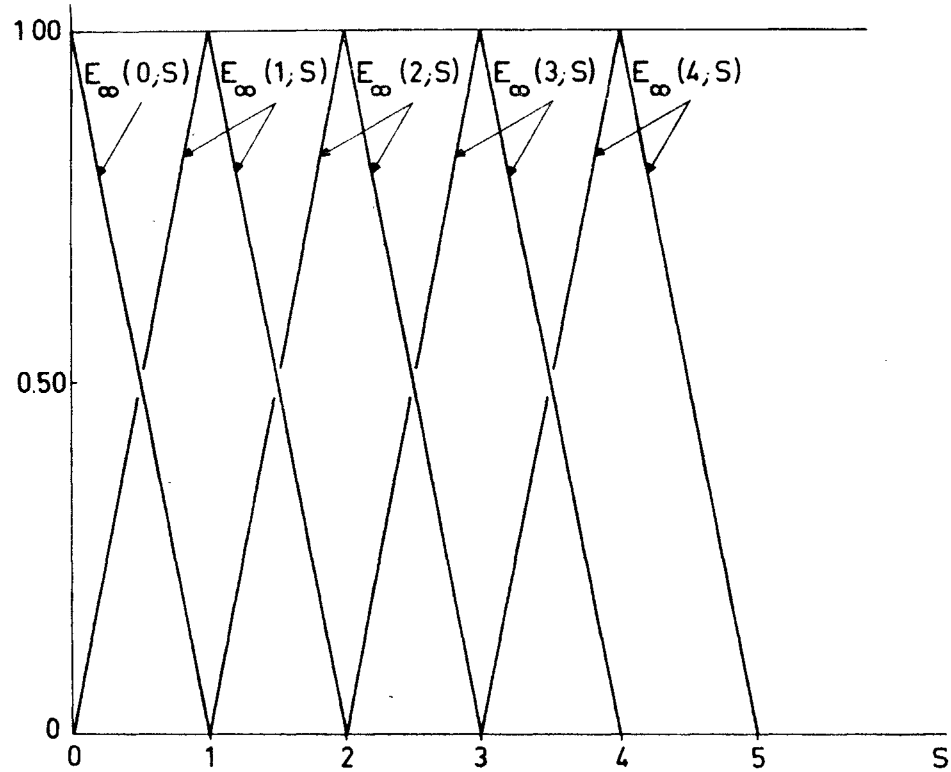


FIG. 5. The $E_\infty(n;\, S)$ for $n = 0, 1, 2, \ldots$ This trivial case corresponds to equally spaced levels.

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## Appendix A

The determinant in eqn. (3.30) is of the form det $[\delta_{ij}-u_iv_j]$. To evaluate such a determinant it is convenient to border it with an extra row and column. Thus

$$\begin{vmatrix} 1-u_1v_1 & -u_1v_2 & \dots & -u_1v_n \\ -u_2v_1 & 1-u_2v_2 & \dots & -u_2v_n \\ \cdots & \cdots & \cdots & \cdots \\ -u_nv_1 & -u_nv_2 & \dots & 1-u_nv_n \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ u_1 & 1-u_1v_1 & -u_1v_2 & \dots & -u_1v_n \\ u_2 & -u_2v_1 & 1-u_2v_2 & \dots & -u_2v_n \\ \vdots & \vdots & & & \\ u_n & -u_nv_1 & -u_nv_2 & \dots & 1-u_nv_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & v_1 & v_2 & \dots & v_n \\ u_1 & 1 & 0 & \dots & 0 \\ u_2 & 0 & 1 & \dots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_n & 0 & 0 & \dots & 1 \end{vmatrix} = 1-\sum_{j=1}^{n} u_jv_j .$$

The cofactor $b_{ij}$ of the element $(i, j)$ in det $[\delta_{ij}-u_iv_j]$ is easy to calculate by the same bordering

$$b_{ij} = \delta_{ij}\left\{1-\sum_{l=1}^{n} u_lv_l\right\}+u_jv_i .$$

## Appendix B

First few terms in the power series expansions of $\lambda_i, f_i(x), E_1(n; S), E_2(n; S), p_1(n; S)$ and $p_2(n; S)$.

$$\lambda_0 = S-\frac{\pi^2}{36}S^3+\frac{23\pi^4}{32400}S^5-\frac{79\pi^6}{5715360}S^7+O(S^9)$$

$$\lambda_1 = \frac{\pi^2}{36}S^3-\frac{\pi^4}{1200}S^5+\frac{41\pi^6}{1470000}S^7+O(S^9)$$

$$\lambda_2 = \frac{\pi^4}{8100}S^5-\frac{\pi^6}{2857680}S^7+O(S^9)$$

$$\lambda_3 = \frac{\pi^6}{4410000}S^7+O(S^9)$$

$$\lambda_i = O(S^{2i+1})$$

$$f_0(x) = (\tfrac{1}{2})^{\frac{1}{2}}\left\{\left(1-\frac{\pi^4}{12960}S^4\right)+\left(-\frac{\pi^2}{36}S^2+\frac{\pi^4}{4536}S^4\right)P_2(x)+\frac{\pi^4}{8400}S^4P_4(x)+O(S^6)\right\}$$

$$f_1(x) = (\tfrac{3}{2})^{\frac{1}{2}} \left\{ \left(1 - \frac{3\pi^4}{140000} S^4\right) P_1(x) + \left(-\frac{\pi^2}{100} S^2 - \frac{\pi^4}{45000} S^4\right) P_3(x) + \frac{\pi^4}{35280} S^4 P_5(x) + O(S^6) \right\}$$

$$f_2(x) = (\tfrac{5}{2})^{\frac{1}{2}} \left\{ \left(\frac{\pi^2}{180} S^2 - \frac{\pi^4}{22680} S^4\right) + \left(1 - \frac{545\pi^4}{6223392} S^4\right) P_2(x) + \left(-\frac{3\pi^2}{490} S^2 - \frac{23\pi^4}{384160} S^4\right) P_4(x) + O(S^6) \right\}$$

$$f_j(x) = \left(\frac{2j+1}{2}\right)^{\frac{1}{2}} P_j(x) + O(S^2)$$

$$E_1(0\,;S) = 1 - S + \frac{\pi^2}{36} S^3 - \frac{\pi^4}{1200} S^5 + \frac{\pi^4}{8100} S^6 + \frac{\pi^6}{70560} S^7 - \frac{\pi^6}{264600} S^8 + \dots$$

$$E_1(1\,;S) = S - \frac{\pi^2}{18} S^3 + \frac{\pi^4}{600} S^5 - \frac{\pi^4}{8100} S^6 - \frac{\pi^6}{35280} S^7 + \frac{\pi^6}{264600} S^8 + \dots$$

$$E_1(2\,;S) = \frac{\pi^2}{36} S^3 - \frac{\pi^4}{1200} S^5 - \frac{\pi^4}{8100} S^6 + \dots$$

$$E_2(0\,;S) = 1 - S + \frac{\pi^2}{36} S^4 - \frac{\pi^4}{675} S^6 + \frac{\pi^6}{17640} S^8 + \dots$$

$$E_2(1\,;S) = S - \frac{\pi^2}{18} S^4 + \frac{2\pi^4}{675} S^6 + \dots$$

$$E_2(2;\,S) = \frac{\pi^2}{36} S^4 - \frac{\pi^4}{675} S^6 + \dots$$

$$p_1(0;\,S) = \frac{\pi^2}{6} S - \frac{\pi^4}{60} S^3 + \frac{\pi^4}{270} S^4 + \frac{\pi^6}{1680} S^5 + \dots$$

$$p_1(1;\,S) = \frac{\pi^4}{270} S^4 - \frac{\pi^6}{4725} S^6 + \dots$$

$$p_1(2;\,S) = \frac{\pi^8}{1764000} S^8 + \dots$$

$$p_2(0;\,S) = \frac{\pi^2}{3} S^2 - \frac{2\pi^4}{45} S^4 + \frac{\pi^6}{315} S^6 + \dots$$

$$p_2(1;\,S) = \frac{\pi^6}{4050} S^7 + \dots$$

$$p_2(2;\,S) = \frac{\pi^{12}}{5358150000} S^{14} + \dots$$

## Appendix C

$$E_4(n\,;\tfrac{1}{2}S) = \prod_{l=0}^{\infty} (1-\lambda_{2l}) \sum_{(i)} \frac{\lambda_{2i_1}}{1-\lambda_{2i_1}} \cdots \frac{\lambda_{2i_n}}{1-\lambda_{2i_n}} \times \left\{ 1 + \tfrac{1}{2} \sum_{j \neq (i)} \frac{\lambda_{2j} b_j}{1-\lambda_{2j}} - \tfrac{1}{2} \left( \sum_{k=i}^{n} b_{i_k} \right) \left( 1 + \sum_{j \neq (i)} \frac{\lambda_{2j} b_j}{1-\lambda_{2j}} \right) \right\},$$

where

$$b_j = f_{2j}(1) \int_{-1}^{1} f_{2j}(x)\, dx.$$

If we introduce roots $\nu_j$ of the equation

$$\tfrac{1}{2} \sum_{i=1}^{\infty} \frac{\lambda_{2i} b_i}{\lambda_{2i} - \nu} = 1.$$

Then

$$E_4(n;\, \tfrac{1}{2}S) = \prod_{l=0}^{\infty} (1-\nu_l) \sum_{(i)} \frac{\nu_{i_1}}{1-\nu_{i_1}} \cdots \frac{\nu_{i_n}}{1-\nu_{i_n}}.$$

The summations (i) are taken over all integers satisfying $0 \le i_1 < i_2 < \ldots < i_n$.