A^*-FUNCTION (I)

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In the present paper we have defined a new generalized function of two variables, viz. $A^*\binom{x}{y}$. The conditions of convergence have been given in detail. The present function includes $S\binom{x}{y}$ and $G\binom{x}{y}$ as particular cases and thus includes a large number of special functions of one and more variables as particular cases.

1. Introduction

Hypergeometric functions of two variables have been studied by Appell and Kampé de Fériet (1926), Baily (1935), Burchnall and Chaundy (1940). Erdélyi (1951) gave the general form of the hypergeometric series of two variables of Appell's type. Agrawal (1965) and Sharma (1965) defined the functions $G(x, y)$ and $S(x, y)$ respectively of two variables. In the present paper we have defined the function $A^*(x, y)$ of two variables which gives $G(x, y)$, $S(x, y)$ and other known functions as particular cases by specializing the parameters.

Define the generalized function $A^*(x, y)$ as

$$A^{*\,m_1,\,0;\,m_2,\,n_2;\,m_3,\,n_3}_{\;\,p_1,\,q_1;\,p_2,\,q_2;\,p_3,\,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\,[((a_{p_1},\alpha_{p_1}));((b_{q_1},\beta_{q_1}))]:\{((c_{p_2},\gamma_{p_2}));((d_{q_2},\delta_{q_2}))\}:[((e_{p_3},\lambda_{p_3}));((f_{q_3},\mu_{q_3}))]\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\psi(s+t)\,\phi(s,t)x^s y^t\,ds\,dt \qquad \ldots \quad (1.1)$$

where

$$\psi(s+t)=\frac{\prod_{j=1}^{m_1}\Gamma(a_j+\alpha_j s+\alpha_j t)}{\prod_{j=m_1+1}^{p_1}\Gamma(1-a_j-\alpha_j s-\alpha_j t)\prod_{j=1}^{q_1}\Gamma(b_j+\beta_j s+\beta_j t)} \qquad \ldots \quad (1.2)$$

$$\phi(s,t)=\frac{\prod_{j=1}^{m_2}\Gamma(1-c_j+\gamma_j s)\prod_{j=1}^{n_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{m_3}\Gamma(1-e_j+\lambda_j t)\prod_{j=1}^{n_3}\Gamma(f_j-\mu_j t)}{\prod_{j=m_2+1}^{p_2}\Gamma(c_j-\gamma_j s)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\prod_{j=m_3+1}^{p_3}\Gamma(e_j-\lambda_j t)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+\mu_j t)} \qquad \ldots \quad (1.3)$$

$$((a_{p_1}, \alpha_{p_1})) = (a_1, \alpha_1), (a_2, \alpha_2) \ldots (a_{p_1}, \alpha_{p_1}). \quad \ldots \quad (1.4)$$

α's, β's, γ's, δ's, λ's and μ's are all positive. Also L_1 and L_2 are contours in the s and t planes respectively running from $-i\infty$ to $+i\infty$ with loops if necessary to ensure that the poles of $\Gamma(d_j-\delta_j s)$, $j = 1, 2, \ldots n_2$, lie to the right and poles of $\Gamma(1-c_j+\gamma_j s)$, $j = 1, 2, \ldots m_2$; $\Gamma(a_j+\alpha_j s+\alpha_j t)$, $j = 1, 2, \ldots m_1$, lie to the left of the contour L_1. The poles of $\Gamma(f_j-\mu_j t)$, $j = 1, 2, \ldots n_3$, lie to the right of L_2 and the poles of $\Gamma(1-e_j+\lambda_j t)$, $j = 1, 2, \ldots m_3$. $\Gamma(a_j+\alpha_j s+\alpha_j t)$, $j = 1, 2, \ldots m_1$, lie to the left of the contour L_2. Further the positive integers $p_1, p_2, p_3, m_1, m_2, m_3, q_1, q_2, q_3, n_2, n_3$ satisfy the following inequalities:

$$q_2, q_3 \geqslant 1, \; p_1, q_1 \geqslant 0, \; 0 \leqslant m_1, m_2, n_2, m_3, n_3 \leqslant p_1, p_2, q_2\, p_3, q_3$$

$$p_1+p_2 \leqslant q_1+q_2, \; p_1+p_3 \leqslant q_1+q_3$$

$$\alpha p_1+\gamma p_2 \leqslant \beta q_1+\delta q_2, \; \alpha p_1+\lambda p_3 \leqslant \beta q_1+\mu q_3$$

where $0 \leqslant m_1, m_2, \ldots n_3 \leqslant p_1, p_2, \ldots, q_3$ means the set of inequalities $0 \leqslant m_1 \leqslant p_1$, $0 \leqslant m_2 \leqslant p_2, \ldots$, etc., and greatest of $\alpha_j, \beta_j, \gamma_j, \delta_j, \lambda_j, \mu_j$ are $\alpha, \beta, \gamma, \delta, \lambda, \mu$. The values of $x = 0$ and $y = 0$ are excluded.

Proceeding identically as for MacRobert (1962) E-function [*see also* Chaturvedi (1970)], we find that $A^*(x, y)$ is an analytic function of x, y provided that

$$|\arg x| < \left(\alpha m_1+\gamma m_2+\delta n_2-\frac{\alpha p_1}{2}-\frac{\gamma p_2}{2}-\frac{Bq_1}{2}-\frac{\delta q_2}{2}\right)\pi \quad \ldots \quad (1.5)$$

$$2(\alpha m_1+\gamma m_2+\delta n_2) > \alpha p_1+\gamma p_2+\beta q_1+\delta q_2 \quad \ldots \quad (1.6)$$

$$|\arg y| < \left(\alpha m_1+\lambda m_3+\mu n_3-\frac{\alpha p_1}{2}-\frac{\lambda p_3}{2}-\frac{\beta q_1}{2}-\frac{\mu q_3}{2}\right)\pi \quad \ldots \quad (1.7)$$

$$2(\alpha m_1+\lambda m_3+\mu n_3) > \alpha p_1+\lambda p_3+\beta q_1+\mu q_3 . \quad \ldots \quad (1.8)$$

2. Finite Series

The following known results have been used in the next derivations:

$$\prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha+r-i}{m}\right) = m^{-r}(\alpha-m+1)_r \prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha-i}{m}\right) \quad \ldots \quad (2.1)$$

$$\prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha-r+i}{m}\right) = (-m)^r[(1-\alpha)_r]^{-1} \prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha+i}{m}\right) \quad \ldots \quad (2.2)$$

where m and r are positive integers.

$$F\left(\begin{matrix} a, \beta, \gamma \\ \rho, \sigma \end{matrix}; 1\right) = \frac{\Gamma(\rho)\,\Gamma(\alpha-\sigma+1)\,\Gamma(\beta-\sigma+1)\,\Gamma(\gamma-\sigma+1)}{\Gamma(1-\sigma)\,\Gamma(\rho-\alpha)\,\Gamma(\rho-\beta)\,\Gamma(\rho-\gamma)} \quad \ldots \quad (2.3)$$

where $\alpha+\rho = \alpha+\beta+\gamma+1$ and one of the parameters α, β, γ is a negative integer.

$$F\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \quad \ldots \quad (2.4)$$

where $R(c-a-b) > 0$.

Now we give the following finite series for the function $A^*(x, y)$ of two variables:

$$\text{(i)} \quad \sum_{r=0}^{n} \frac{n_{c_r}(-1)^{n+r}}{\Gamma(1+d_1-c_{p_2}+r)} A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| [((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}, \delta_1) ; (d_1+r, \delta_1), ((d_{2,q_2}, \delta_{2,q_2}))]\right.$$

$$= \frac{1}{\Gamma(1+d_1-c_{p_2}+n)} A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2-n}, \delta_1)\right] \quad \ldots \quad \ldots \quad (2.5)$$

where $R(c_{p_2}) < n+1$, n and r are positive integers and the set of conditions (1.5) to (1.8).

PROOF: To prove (2.5), use (1.1) in the L.H.S., change the order of integration and summation which is justified under the conditions stated to obtain

$$\frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(a_j+\alpha_j s+\alpha_j t) \prod_{j=1}^{m_2} \Gamma(1-c_j+\gamma_j s) \prod_{j=1}^{n_2} \Gamma(d_j-\delta_j s) \prod_{j=1}^{m_3} \Gamma(1-e_j+\lambda_j t)}{\prod_{j=m_1+1}^{p_1} \Gamma(1-a_j-\alpha_j s-\alpha_j t) \prod_{j=1}^{q_1} \Gamma(b_j+B_j s+B_j t) \prod_{j=m_2+1}^{p_2} \Gamma(c_j-\gamma_j s) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+\delta_j s)}$$

$$\frac{\prod_{j=1}^{n_3} \Gamma(f_j-\mu_j t)}{\prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+\mu_j t) \prod_{j=m_3+1}^{p_3} \Gamma(e_j-\lambda_j t)} (-1)^n F\left(\begin{matrix} -n, d_1-\delta_1 s \\ 1+d_1-c_{p_2} \end{matrix} ; 1\right) x^s y^t \, ds \, dt. \quad (2.6)$$

Now use (2.4) along with

$$\Gamma(1-c_{p_2}+n+\delta_1 s)\Gamma(c_{p_2}-n-\delta_1 s) = (-1)^n \Gamma(c_{p_2}-\delta_1 s) \, \Gamma(1-c_{p_2}+\delta_1 s) \quad (2.7)$$

and interpret the result with the help of (1.1), we get the R.H.S. of (2.5).

Proceeding in the same way and using the known results (2.1) to (2.4) we obtain the following series for the function $A^*(x, y)$:

$$\text{(ii)} \quad \sum_{r=0}^{n} n_{c_r}(-1)^{n+r} A^* \left[\begin{matrix} x \\ y \end{matrix}\middle| [((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}+r, \delta_1); (d_1+r, \delta_1), ((d_{2,q_2}, \delta_{2,q_2}))]\right]$$

$$= \frac{\Gamma(1-c_{p_2}+d_1)}{\Gamma(1+d_1-c_{p_2}-n)} A^* \left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}+n, \delta_1); ((d_{q_2}, \delta_{q_2}))\right] \quad (2.8)$$

where $R(d_1-c_{p_2}) < n$, n and r are positive integers and the set of conditions (1.5) to (1.8).

$$\text{(iii)} \quad \sum_{r=0}^{n} n_{c_r} A^* \left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}+r, \delta_1); ((d_{q_2-1}, \delta_{q_2-1})), (d_{q_2}+r, \delta_1)\right]$$

$$= \frac{\Gamma(c_{p_2}-d_{q_2}+n)}{\Gamma(c_{p_2}-d_{q_2})} A^* \left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}+n, \delta_1); ((d_{q_2-1}, \delta_{q_2-1})), (d_{q_2}, \delta_1)\right]$$

$$\ldots \quad (2.9)$$

where $R(c_{p_2}-d_{q_2}) < -n$, n and r are positive integers and the set of conditions (1.5) to (1.8).

$$\text{(iv)} \sum_{r=0}^{n} \frac{n_{c_r}(-1)^{n+r}}{\Gamma(1-c_1+d_{q_2}+r)} A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| (c_1-r, \gamma_1), ((c_{2,p_2}, \gamma_{2,p_2})); ((d_{q_2-1}, \delta_{q_2-1})), (d_{q_2}, \gamma_1)\right]$$

$$= \frac{1}{\Gamma(1-c_1+d_{q_2}+n)} A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2}, \gamma_{p_2})); ((d_{q_2-1}, \delta_{q_2-1})), (d_{q_2}+n, \gamma_1)\right] \quad (2.10)$$

where $R(d_{q_2}+n) > 0$, n and r are positive integers and set of conditions (1.5) to (1.8).

3. Particular Cases

Here we give only the particular cases of (2.5). The particular cases of the other results can be obtained in the same way.

(i) For $n = 1$, (2.5) reduces to

$$(1-d_1-c_{p_2})A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}, \delta_1)\right]$$

$$= A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2}, \delta_1); (d_1+1, \delta_1), ((d_{2,q_2}, \delta_{2,q_2}))\right]$$

$$-A^*\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1}, \gamma_{p_2-1})), (c_{p_2-1}, \delta_1); ((d_{q_2}, \delta_{q_2}))\right]. \quad .. \quad (3.1)$$

(ii) If we substitute $\alpha_j = \beta_j = \gamma_j = \delta_j = \lambda_j = \mu_j = 1$ in (3.1) then we get a recurrence relation in S-function (Sharma 1965),

$$(1-d_1-c_{p_2})\, S\binom{x}{y} = S\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2})); d_1+1, ((d_{2,q_2}))\right] - S\left[\begin{matrix} x \\ y \end{matrix}\middle| ((c_{p_2-1})), c_{p_2-1}; ((d_{q_2}))\right] \quad .. \quad (3.2)$$

which on further specializing the parameters will give similar relation in Agrawal's $G(x, y)$ and consequently in Meijer's G-function which is itself a generalized function of one variable.

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