SOME STEADY STATE THERMOELASTIC STRESS DISTRIBUTION IN THE VICINITY OF AN EXTERNAL CRACK IN AN INFINITE MEDIUM

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Some steady state thermal stress distribution in an infinite medium containing a stress-free flat external crack is considered. The case of temperature distribution which is even with respect to the plane of the crack is treated. The displacement vector is represented by means of harmonic functions, one of which is directly related to the temperature field, and the imposition of the elastic conditions on the crack leads to a mixed boundary value problem which is solved using standard technique. Simple expression is given for stress intensity factor.

1. Introduction

In this paper we treat some steady state thermoelastic problem for an infinite elastic solid containing an external crack whose faces are stress free and with temperature distribution which is even with respect to the plane of the crack. The complimentary problem of stress distributions in an infinite solid due to the application of normal pressure to the faces of external crack has been considered by Lowengrub (1966). The three-dimensional analogue of the problem has been considered by Shail (1969).

The general approach of the paper is to represent appropriate solutions of the linear thermoelastic equations in terms of harmonic functions. In section 2 the basic representation of the displacement vector is given in terms of the harmonic functions, one of which is directly related to the temperature field. In section 3, even temperature fields are considered and the problem is reduced to the solution of a pair of dual integral equations.

Finally, in section 4, the effect of a special type of temperature field is considered and the results are illustrated numerically.

2. Basic Solutions of the Thermoelastic Equations

We consider an infinite, isotropic, elastic medium which contains a flat crack (Fig. 1) whose faces are stress free. Using cartesian coordinates (x, y), the faces of the crack are described by the relations $y = 0$, $x \geqslant 1$, with a suitable choice of unit of length. There is established in the medium a steady temperature field $T(x, y)$ where T is the deviation of the absolute temperature from the temperature of the medium in a state of zero stress and strain. In the absence of body forces or heat sources within the medium, the steady state equations of classical thermoelasticity are

$$(1-2\eta)\nabla^2 u + \text{grad}\,[\text{div}\, u - 2\alpha(1+\eta)T] = 0 \qquad \ldots \quad \ldots \quad (2.1)$$

and

$$\nabla^2 T = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

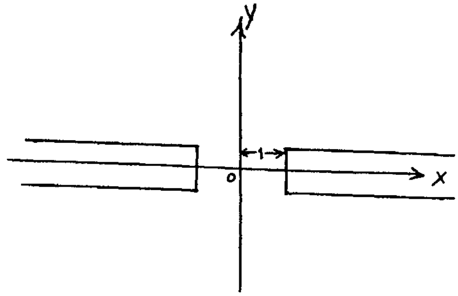

FIG. 1

where u is the displacement vector, η the Poisson's ratio and α the coefficient of linear expansion of the medium. Further the stress vector Y across an element of surface area whose normal is parallel to y-axis is

$$Y = \mu \left\{ \frac{[2\eta \,\text{div}\, u - 2\alpha(1+\eta)T]}{1-2\eta} K + \frac{\partial u}{\partial y} + \text{grad}\,(u\,.\,K) \right\} \qquad \ldots \quad (2.3)$$

where μ is the modulus of rigidity and K is a unit vector parallel to the y-axis.

Consider the case of temperature distribution which is even in y; the elastic problem is symmetrical about $y = 0$ and hence τ_{xy} is identically zero on $y = 0$. An appropriate solution of (2.1) and (2.2) with shear stress on $y = 0$ are in terms of harmonic functions ϕ and χ as

$$u = \left(\frac{\partial \phi}{\partial y} + \beta \frac{\partial \chi}{\partial y}\right) K - \text{grad}\,\{\phi + \tfrac{1}{2}(\beta-1)\chi\} + y\,\text{grad}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \chi}{\partial y}\right) \qquad \ldots \quad (2.4)$$

$$T = \frac{2(1-\eta)}{\alpha(1+\eta)} \frac{\partial^2 \phi}{\partial y^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

with $\beta = 3-4\eta$.

Further, from (2.3) and (2.4),

$$\sigma_y = 2\mu \left\{ y \frac{\partial^3}{\partial y^3}(\phi - \chi) - \frac{\partial^2}{\partial y^2}(\phi - \chi) \right\}. \qquad \ldots \quad \ldots \quad (2.6)$$

From (2.4) and (2.5), ϕ and χ are even functions of y whilst u_y is odd, hence

$$u_y \big|_{y=0} = 0 \quad (0 \le x \le 1). \qquad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

Also all components of U and Y are continuously differentiable at interior points of the medium. At the edge of the crack σ_y can be expected to have a singularity of order $(1-x^2)^{-\frac{1}{2}}$ as $x \to 1-$.

3. Solution of the Problem

Since the temperature distribution is even, we refer to equation (2.4) through (2.7). As the crack faces are stress free, we have

$$\sigma_y\big|_{y=0} = 0 \quad (x > 1). \qquad (3.1)$$

Combining (2.7) and (3.1), it follows that χ is determined from the mixed boundary value problem:

$$\frac{\partial \chi}{\partial y}\bigg|_{y=0} = 0 \quad (0 \leq x < 1) \qquad (3.2)$$

$$\frac{\partial^2 \chi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\alpha(1+\eta)}{2(1-\eta)} T \quad \text{on } y = 0,\ x > 1. \qquad (3.3)$$

Suppose now that temperature field has the form

$$T(x, y) = t(x, y). \qquad (3.4)$$

Representing the odd harmonic $\dfrac{\partial \chi}{\partial y}$ by

$$\frac{\partial \chi}{\partial y} = \pm \int_0^\infty \xi^{-1} B(\xi) \cos(\xi x) e^{-\xi|y|}\, d\xi \quad (y \gtrless 0) \qquad (3.5)$$

it follows from (3.2) and (3.3) that $B(\xi)$ is determined by the dual integral equations:

$$\int_0^\infty \xi^{-1} B(\xi) \cos(\xi x)\, d\xi = 0 \quad (0 \leq x < 1) \qquad (3.6)$$

$$\int_0^\infty B(\xi) \cos(\xi x)\, d\xi = \frac{\alpha(1+\eta)}{2(1-\eta)} t(x, 0) \quad (x > 1). \qquad (3.7)$$

The solution of (3.6) and (3.7) can be written as (Lowengrub 1966)

$$B(\xi) = 2/\pi \int_1^\infty h(x) \sin(\xi x)\, dx + 2/\pi \int_0^1 t^2 J_1(\xi t)\, dt \int_1^\infty \frac{h(x)\, dx}{(x^2-t^2)^{3/2}} \quad (1 \leq t < \infty) \qquad (3.8)$$

where

$$h(x) = \frac{-\alpha(1+\eta)}{2(1-\eta)} t(x, 0) \quad 0 \leq x < \infty,$$

a regular function of x.

From (2.6) and (3.8) it is straightforward to show that

$$\sigma_{y(x,0)} = -\frac{2\mu\alpha(1+\eta)}{\pi(1-\eta)} \sigma(x) \quad 0 \leq x < 1 \qquad (3.9)$$

where $\sigma(x)$ is defined by

$$\sigma(x) = (1-x^2)^{-\frac{1}{2}} \int_1^\infty h(u)\, du + \int_1^\infty \frac{u\sqrt{(u^2-1)}\, h(u)\, du}{(u^2-x^2)\sqrt{(1-x^2)}} \quad .. \; (3.10)$$

where $h(u) = t(u, 0)$.

We can also show that the displacement component u_y is given by the equation

$$u_y(x, y) = -\alpha(1+\eta)/(1-\eta) \int_{-1}^1 \sigma(w)\, U_y(x-w, y)\, dw \quad .. \; (3.11)$$

where Green's function U_y is defined as:

$$U_y(x, y) = \frac{x^2}{x^2+y^2} + \frac{1-\eta}{1-2\eta} \log (x^2+y^2). \quad .. \quad .. \; (3.12)$$

4. A Particular Case

Let

$$t(x, 0) = \frac{(1-\eta)}{\mu(1+\eta)\alpha} P\, \delta(c-x) \quad c > 0. \quad .. \quad .. \; (4.1)$$

Then from (3.10), we have

$$\sigma(x) = \frac{1-\eta}{\mu(1+\eta)\alpha} P(1-x^2)^{-\frac{1}{2}} \left[1 + \frac{c\sqrt{(c^2-1)}}{(c^2-x^2)}\right]. \quad .. \quad .. \; (4.2)$$

Therefore

$$\sigma_y(x, 0) = -2/\pi\, P(1-x^2)^{-\frac{1}{2}} \left[1 + \frac{c\sqrt{(c^2-1)}}{(c^2-x^2)}\right] \quad .. \quad .. \; (4.3)$$

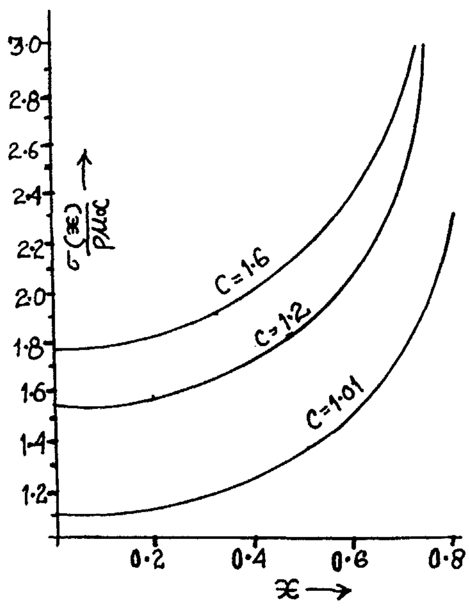

FIG. 2. The variation of $\sigma(x)/\mu\, P\alpha$ with x.

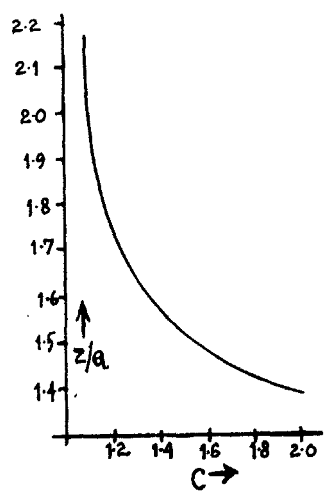

FIG. 3. The variation of stress intensity factor with c.

and

$$\sigma_y(0,\,0) = -2/\pi\, P\left(1+\frac{\sqrt{(c^2-1)}}{c}\right). \qquad \cdots \quad \cdots \quad \cdots \quad (4.4)$$

This expression yields value of the stress intensity factor as:

$$\lim_{x\to 1-} (1-x^2)^{-\frac{1}{2}}\,\sigma_y(x,\,0)$$

$$= -2/\pi\, P\left(1+\frac{c}{\sqrt{(c^2-1)}}\right). \qquad \cdots \quad \cdots \quad \cdots \quad (4.5)$$

Substituting (4.2) into (3.11), we obtain an integral expression for $u_y(x,\,0)$ which can be evaluated by numerical methods.

In Fig. 2 we see the variation of $\sigma(x)/\mu\, P\alpha$ with x for $c = 1{\cdot}01,\ 1{\cdot}2,\ 1{\cdot}6$ and $\eta = \frac{1}{3}$. Fig. 3 represents the variation of stress intensity factor with c.

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