ON MULTIPLE INTEGRALS FOR MEIJER'S G-FUNCTION

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The aim of this paper is to evaluate two multiple integrals involving product of Meijer's G-function. Since the G-function is quite general in character and many special functions appearing in applied mathematics follow as its particular cases, several other integrals involving products of different special functions can be obtained as special cases of our main integrals. The two multiple integrals involving products of modified Bessel functions of the second kind given by Ragab (1965), the generalized Watson function introduced by Bhatnagar (1953) and the integral involving product of two G-functions evaluated by Meijer (1941) follow as particular cases of our main results.

1. INTRODUCTION

Meijer's G-function (Erdélyi 1953, p. 207) has been defined and denoted as:

$$G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right]=G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right]$$

$$=\left(\frac{1}{2\pi i}\right)\int_L\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-s)\prod\limits_{j=1}^{n}\Gamma(1-a_j+s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+s)\prod\limits_{j=n+1}^{p}\Gamma(a_j-s)}x^s\,ds \quad .. \quad (1.1)$$

where L is a suitable contour.

In what follows for the sake of brevity $a_{n_i}+\frac{k_i}{\sigma}$ stands for the $\sum\limits_{i=1}^{S} n_i$ parameters: $a_{n_1}+\frac{k_1}{\sigma}, \ldots, a_{n_S}+\frac{k_S}{\sigma}$; where $a_{n_1}+\frac{k_i}{\sigma}$ denotes n_1 parameters: $a_{1_1}+\frac{k_1}{\sigma}, \ldots, a_{n_1}+\frac{k_1}{\sigma}$; $b_{m_i}+\frac{k_i}{\sigma}$ stands for the $\sum\limits_{i=1}^{S} m_i$ parameters $b_{m_1}+\frac{k_1}{\sigma}, \ldots, b_{m_S}+\frac{k_S}{\sigma}$; where $b_{m_1}+\frac{k_1}{\sigma}$ denotes m_1 parameters: $b_{1_1}+\frac{k_1}{\sigma}, \ldots, b_{m_1}+\frac{k_1}{\sigma}$. $\left(a_{n_i}+\frac{k_i}{\sigma}, a_{p_i}\right)$ stands for the $\sum\limits_{i=1}^{S}(p_i-n_i)$ parameters: $\left(a_{n_1+1}+\frac{k_1}{\sigma}, \ldots, a_{p_1}+\frac{k_1}{\sigma}\right), \ldots, \left(a_{n_S+1}+\frac{k_S}{\sigma}, \ldots, a_{p_S}+\frac{k_S}{\sigma}\right)$; $\left(b_{m_i}+\frac{k_i}{\sigma}, b_{q_i}\right)$ stands for the $\sum\limits_{i=1}^{S}(q_i-n_i)$ parameters: $\left(b_{m_1+1}+\frac{k_1}{\sigma}, \ldots, b_{q_1}+\frac{k_1}{\sigma}\right), \ldots, \left(b_{m_S+1}+\frac{k_S}{\sigma}, \ldots, b_{q_S}+\frac{k_S}{\sigma}\right)$. $1-a_p$ and $1-b_q$ stand for the p and q parameters: $1-a_1, \ldots, 1-a_p$ and $1-b_1, \ldots, 1-b_q$ respectively.

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2. Integrals

Our main integrals are:

$$\prod_{r=1}^{S}\int_0^\infty x_r^{k_r-1}G_{p_r,q_r}^{m_r,n_r}\left[zx_r^\sigma\middle|\begin{matrix}a_{p_r}\\b_{q_r}\end{matrix}\right]G_{p,q}^{m,n}\left[ay^\rho(x_1\ldots x_S)^\sigma\middle|\begin{matrix}a_p\\b_q\end{matrix}\right]dx_r=\sigma^{-S}(z)^{-\left(\frac{1}{\sigma}\right)\sum_{r=1}^{S}(k_r)}$$

$$\times G_{P+q,Q+p}^{M+n,N+m}\left[\frac{z^S}{ay^\rho}\middle|\begin{matrix}a_{n_i}+\frac{k_i}{\sigma},\ 1-b_q,\ \left(a_{n_i}+\frac{k_i}{\sigma},\ a_{p_i}\right)\\ b_{m_i}+\frac{k_i}{\sigma},\ 1-a_p,\ \left(b_{m_i}+\frac{k_i}{\sigma},\ b_{q_i}\right)\end{matrix}\right] \quad \ldots (2.1)$$

provided that

$R(k_r+\sigma(b_j+b_i))>0(j=1_r,\ldots,m_r(r=1,\ldots,S);\ i=1,\ldots,m)$;
$R(k_r+\sigma(a_h+a_{h'}-2))<0(h=1_r,\ldots,n_r(r=1,\ldots,S);\ h'=1,\ldots,n)$;
$|\arg z|<(m_r+n_r-1/2\,p_r-1/2\,q_r)\pi,\ (m_r+n_r-1/2\,p_r-1/2\,q_r)>0$;
$|\arg a|<(m+n-1/2\,p-1/2\,q)\pi,\ (m+n-1/2\,p-1/2\,q)>0$;

M, N, P, Q stand for the quantities $\sum_{r=1}^{S}(m_r)$, $\sum_{r=1}^{S}(n_r)$, $\sum_{r=1}^{S}(p_r)$, and $\sum_{r=1}^{S}(q_r)$ respectively, S is a positive integer ($S=1, 2, 3, \ldots$), $k_r \geqslant 0$ for every value of r, y is real and positive, $\sigma>0$ and $\rho>0$.

$$\prod_{r=1}^{S}\int_0^\infty x_r^{k_r-1}G_{p_r,q_r}^{m_r,n_r}\left[zx_r^\sigma\middle|\begin{matrix}a_{p_r}\\b_{q_r}\end{matrix}\right]G_{p,q}^{m,n}\left[\frac{ay^\rho}{(x_1\ldots x_S)^\sigma}\middle|\begin{matrix}a_p\\b_q\end{matrix}\right]dx_r=\sigma^{-S}(z)^{-\left(\frac{1}{\sigma}\right)\sum_{r=1}^{S}(k_r)}$$

$$\times G_{P+p,Q+q}^{M+m,N+n}\left[z^Say^\rho\middle|\begin{matrix}a_{n_i}+\frac{k_i}{\sigma},\ a_p,\ \left(a_{n_i}+\frac{k_i}{\sigma},\ a_{p_i}\right)\\ b_{m_i}+\frac{k_i}{\sigma},\ b_q,\ \left(b_{m_i}+\frac{k_i}{\sigma},\ b_{q_i}\right)\end{matrix}\right] \quad \ldots (2.2)$$

provided that

$R(k_r+\sigma(b_j-b_i))>0(j=1_r,\ldots,m_r(r=1,\ldots,S);\ i=1,\ldots,m)$;
$R(k_r+\sigma(a_h-a_{h'}-2))<0(h=1_r,\ldots,n_r(r=1,\ldots,S);\ h'=1,\ldots,n)$;
$|\arg z|<(m_r+n_r-1/2\,p_r-1/2\,q_r)\pi,\ (m_r+n_r-1/2\,p_r-1/2\,q_r)>0$;
$|\arg a|<(m+n-1/2\,p-1/2\,q)\pi,\ (m+n-1/2\,p-1/2\,q)>0$;

$S=1, 2, 3, \ldots$; M, N, P, Q stand for the quantities defined in (2.1), $\sigma>0$, $\rho>0$, $k_r\geqslant 0$ for every value of r.

Proof: Expanding the left-hand side of (2.1), we have

$$\int_0^\infty x_S^{k_S-1}G_{p_S,q_S}^{m_S,n_S}\left[zx_S^\sigma\middle|\begin{matrix}a_{p_S}\\b_{q_S}\end{matrix}\right].\int_0^\infty x_{S-1}^{k_{S-1}-1}G_{p_{S-1},q_{S-1}}^{m_{S-1},n_{S-1}}\left[zx_{S-1}^\sigma\middle|\begin{matrix}a_{p_{S-1}}\\b_{q_{S-1}}\end{matrix}\right]$$

$$\ldots\int_0^\infty x_1^{k_1-1}G_{p_1,q_1}^{m_1,n_1}\left[zx_1^\sigma\middle|\begin{matrix}a_{p_1}\\b_{q_1}\end{matrix}\right].\,G_{p,q}^{m,n}\left[ay^\rho(x_1\ldots x_S)^\sigma\middle|\begin{matrix}a_p\\b_q\end{matrix}\right]dx_S\ldots dx_1. \quad (2.3)$$

Now integrating the last integral with respect to x_1 with the help of the result given in Erdélyi (1954, p. 422 (14)) and substituting the result thus obtained in (2.3) and then integrating with respect to x_2 and so on. Repeating this procedure up to S-times, we arrive at the desired result.

The result (2.2) can be established similarly.

3. Particular Cases

(a) Taking

$$m_1 = m_2 = \ldots = m_S = 2, \quad n_1 = n_2 = \ldots = n_S = 0, \quad n = p = 0,$$

$$a = z = 1/4, \quad q_1 = q_2 = \ldots = q_S = 2, \quad p_1 = p_2 = \ldots = p_S = 0,$$

$$m = q = 2, \quad \rho = \sigma = 2, \quad b_{1_1} = \frac{\nu_1}{2}, \; b_{1_2} = \frac{\nu_2}{2}, \ldots, \; b_{1_S} = \frac{\nu_S}{2}, \; b_1 = \frac{\mu}{2},$$

$$b_2 = -\frac{\mu}{2}, \; b_{2_1} = -\frac{\nu_1}{2}, \; b_{2_2} = -\frac{\nu_2}{2}, \ldots, \; b_{2_S} = -\frac{\nu_S}{2}$$

in (2.1) and (2.2) and using the results given in Erdélyi (1953, p. 216 (4)) and

$$G_{p,q}^{m,n}\left[x \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right] = \sum_{h=1}^{n} \frac{\prod\limits_{J=m+1}^{q} \sin(a_h - b_j)\pi \; \pi^{m+n-q-1}}{\prod\limits_{\substack{i=1 \\ i \neq h}}^{n} \sin(a_h - a_i)\pi} x^{a_h - 1}$$

$$\times E\left[\begin{matrix} 1+b_1-a_h, \ldots, 1+b_q-a_h \\ 1+a_1-a_h, \ldots, 1+a_p-a_h \end{matrix}; (-1)^{m+n-q-1}x\right] \qquad (3.1)$$

respectively and then replacing S by $m-1$, we get the results:

$$\prod_{r=1}^{m-1} \int_0^\infty x_r^{k_r-1} K_{\nu_r}(x_r) dx_r \,.\, K_\mu(y \,.\, x_1 \ldots x_{m-1}) = 2^{\sum\limits_{r=1}^{m-1}(k_r) - 2m+1} \pi \sum_{\mu, -\mu} \left(\frac{1}{\sin \mu\pi}\right)$$

$$\times (2^{2m-4}x^2)^{-\frac{\mu}{2}} E\left[\frac{k_1 \pm \nu_1 - \mu}{2}, \ldots, \frac{k_{m-1} + \nu_{m-1} - \mu}{2} : \frac{e^{\pm i\pi}}{(x^2 2^{2m-4})}\right] \quad \ldots \quad (3.2)$$

provided that $R(k_r \pm \nu_r \pm \mu) > 0$, $r = 1, 2, \ldots, m-1$, where $m = 2, 3, \ldots, y$ is real and positive and the $\sum\limits_{\mu, -\mu}$ means that the expression following it μ is to be replaced by $-\mu$ and the two expressions are then added; and

$$\prod_{r=1}^{m-1} \int_0^\infty x_r^{k_r-1} K_{\nu_r}(x_r)\, dx_r \,.\, K_\mu\left(\frac{y}{x_1 \ldots x_{m-1}}\right) = 2^{\sum\limits_{r=1}^{m-1}(k_r) - 2m+1} \left(\frac{1}{2\pi}\right) \sum_{i, -i} (1/i)$$

$$\times E\left[\frac{k_1 \pm \nu_1}{2}, \ldots, \frac{k_{m-1} \pm \nu_{m-1}}{2}, \pm\frac{\mu}{2}, 1 : \frac{e^{i\pi}}{2^{2m}x^2}\right] \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where $m = 2, 3, \ldots$ and $\sum\limits_{i, -1}$ is same as defined above,

recently given by Ragab (1965).

Bhatnagar (1953) has given the generalization of Watson function $\tilde{\omega}_{\mu,\nu}(x)$ as:

$$\tilde{\omega}_{\mu_1, \ldots, \mu_n}(x) = \sqrt{x} \int_0^\infty \ldots \int_0^\infty J_{\mu_1}(x_1) \,.\, J_{\mu_2}(x_2) \ldots J_{\mu_{n-1}}(x_{n-1})$$

$$\times J_{\mu_n}\left(\frac{x}{x_1 \ldots x_{n-1}}\right) \,.\, (x_1 \ldots x_{n-1})^{-1}\, dx_1 \ldots dx_{n-1} \quad \ldots \quad (3.4)$$

where $R(\mu_k+1/2) \geqslant 0$, $k = 1, 2, \ldots, n$ and μ's may be permuted among themselves.

(b) Now if we take

$$m_1 = m_2 = \ldots = m_S = 1,\quad n_1 = n_2 = \ldots = n_S = 0,\quad n = p = 0,$$

$$a = z = 1/4,\quad q_1 = q_2 = \ldots = q_S = 2,\quad p_1 = p_2 = \ldots = p_S = 0,\quad m = 1,$$

$$q = 2,\ \sigma = \rho = 2,\ b_{1_1} = \frac{\mu_1}{2},\ b_{1_2} = \frac{\mu_2}{2}, \ldots, b_{1_S} = \frac{\mu_S}{2},$$

$$b_{2_1} = -\frac{\mu_1}{2},\ b_{2_1} = -\frac{\mu_2}{2}, \ldots, b_{2_S} = -\frac{\mu_S}{2},\ b_1 = \frac{\mu}{2},\ b_2 = -\frac{\mu}{2},$$

and $k_r = 0 (r = 1, \ldots, S)$,

in (2.2) and use the result given in Erdélyi (1953, p. 216 (3)), we get

$$\prod_{r=1}^{S} \int_0^\infty x_r^{-1} J_{\mu_r}(x_r) J_\mu\left(\frac{y}{x_1 \ldots x_S}\right) dx_r$$

$$= 2^{-S} G^{S+1,\,0}_{0,\,2S+2}\left[\frac{y^2}{2^{2S+2}}\,\middle|\,\frac{\mu_1}{2}, \ldots, \frac{\mu_S}{2}, \pm\frac{\mu}{2}, -\frac{\mu_1}{2}, \ldots, -\frac{\mu_S}{2}\right]. \qquad (3.5)$$

Now multiplying both the sides of (3.5) by $\sqrt{x}$, taking $S = n-1$, replacing μ by μ_n, and then comparing (3.5) with (3.4), we have

$$\tilde{\omega}_{\mu_1, \ldots, \mu_n}(x) = 2^{1-n}\sqrt{x}\, G^{n,\,0}_{0,\,2n}\left[\frac{x^2}{2^{2n}}\,\middle|\,\frac{\mu_1}{2}, \ldots, -\frac{\mu_n}{2}, -\frac{\mu_1}{2}, \ldots, -\frac{\mu_n}{2}\right], \qquad (3.6)$$

a result which expresses the generalized Watson function in terms of Meijer's G-function.

(c) Lastly, if we take $S = 1$, $k_1 = 1$, $\sigma = \rho = 1$ and $y = 1$, then we get a well-known result given by Meijer (1941, p. 84, eqn. (17)).

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