AXISYMMETRIC STAGNATION POINT FLOW WITH UNIFORM SUCTION

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The momentum integral and the kinetic energy integral equations for axisymmetric boundary layers with suction have been used with the aid of a singly infinite family of velocity profiles to study the rotationally symmetrical stagnation point flow with uniform suction. The results obtained have been found to be better than those obtained by Schlichting by the use of the momentum integral equation and wall compatibility condition.

1. INTRODUCTION

Homann (1936) and Frössling (1940) first obtained the exact solution of the Navier-Stokes equations for the rotationally symmetrical stagnation point flow and found that the boundary layer thickness was independent of the distance along the wall and the velocity profiles were similar.

Schlichting (1948) used the momentum integral equation to calculate the axisymmetric stagnation point flow with suction. Truckenbrodt (1956) made an estimate of the function in the momentum integral equation and suggested a simplified method for the calculation of the momentum thickness.

In the present paper the momentum and the kinetic energy integral equations have been used to calculate the boundary layer for the rotationally symmetrical stagnation point flow with uniform suction. The value of the momentum thickness parameter for a solid wall problem obtained by the present method is seen to be more accurate than the value obtained by the other known methods and it is expected that the results for a porous wall problem too obtained by the present method would be the most satisfactory.

2. Momentum Integral Equation

For steady axisymmetric incompressible flow the boundary layer equation and the equation of continuity are:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad \text{(1)}$$

$$\frac{\partial(ur)}{\partial x}+\frac{\partial(vr)}{\partial y}=0. \qquad \text{(2)}$$

Here x is the coordinate along the meridian section, y the coordinate normal to the surface and u, v the velocities along x, y respectively. $r(x)$ is the radius of cross-section at right angles to the axis.

With continuous suction at the surface the boundary conditions are

$$\left.\begin{array}{l} y=0 : u=0. \quad v=v_s \\ y=\infty : u=U(x) \end{array}\right\} \qquad \text{(3)}$$

where v_s is the normal velocity at the surface and $U(x)$ is the potential flow velocity.

Substituting for $v=v_s-\frac{1}{r}\int_0^y \frac{\partial(ur)}{\partial x}\,dy$

from eqn. (2) into eqn. (1) and integrating across the boundary layer, we get

$$\frac{d\theta}{dx}=\frac{\tau_0}{\rho U^2}-\left(2+\frac{\delta^*}{\theta}\right)\frac{\theta}{U}\frac{dU}{dx}-\frac{\theta}{r}\frac{dr}{dx}+\frac{v_s}{U}$$

where

$$\delta^*=\int_0^\infty\left(1-\frac{u}{U}\right)dy; \quad \theta=\int_0^\infty\frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

and

$$\tau_0(x)=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}.$$

In another form

$$\frac{dt^*}{d\bar{x}}=\frac{2}{\bar{U}}[l-(2+H)\Lambda+\lambda-Z] \qquad \text{(4)}$$

where

$$t^*=\left(\frac{\theta}{a}\right)^2\frac{U_0 a}{\nu}, \quad \bar{x}=\frac{x}{a}, \quad \bar{U}=\frac{U(x)}{U_0}$$

$$l=\frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad H=\frac{\delta^*}{\theta}, \quad Z=\frac{\theta^2}{\nu}\frac{U}{r}\frac{dr}{dx}$$

$$\Lambda=\frac{\theta^2}{\nu}\frac{dU}{dx}, \quad \bar{v}_s=\frac{v_s}{U_0}\sqrt{\frac{U_0 a}{\nu}}$$

and

$$\lambda=\frac{v_s\theta}{\nu}=t^{*\frac{1}{2}}\bar{v}_s.$$

3. Kinetic Energy Integral Equation

Adding

$$\frac{u}{2r}\left\{\frac{\partial(ur)}{\partial x}+\frac{\partial(vr)}{\partial y}\right\}$$

to the left-hand side of eqn. (1), multiplying through by u and integrating across the boundary layer we have

$$\frac{d}{dx}\left(\frac{\epsilon^2}{\nu}\right)=\frac{2H_\epsilon}{U}[2D-(3\Lambda+z)H_\epsilon+\lambda] \quad \ldots \quad (5)$$

where

$$\epsilon=\int_0^\infty \frac{u}{U}\left[1-\left(\frac{u}{U}\right)^2\right]dy\,,\; H_\epsilon=\frac{\epsilon}{\theta}$$

and

$$D=\int_0^\infty \left(\frac{\theta}{U}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 d\left(\frac{y}{\theta}\right).$$

The variation of H_ϵ is given by

$$\frac{dH_\epsilon}{d\bar{x}}=\frac{1}{\bar{U}t^*}[2D-H_\epsilon\{l-(H-1)\Lambda+\lambda\}+\lambda]. \quad \ldots \quad (6)$$

4. Wall Compatibility Condition

At the surface of the body where $u=0$ and $v=v_s$ eqn. (1) becomes

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}=-U\frac{dU}{dx}+v_s\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

i.e.

$$m=-\Lambda+l\lambda \quad \ldots \quad (7)$$

where

$$m=\frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}.$$

5. Family of Velocity Profiles (Schlichting 1949)

The one parameter family of velocity profiles given by Schlichting (1949) for the approximate calculation of the boundary layers with suction is

$$\frac{u}{U}=F_1(\eta)+KF_2(\eta)$$

where

$$\eta=\frac{y}{\delta(x)}$$

$$F_1(\eta)=1-e^{-\eta}$$

$$F_2(\eta)=F_1-\sin\frac{\pi\eta}{6},\quad 0\leq\eta\leq 3$$

and

$$F_2(\eta) = F_1 - 1, \quad \eta \geqslant 3.$$

For this system of velocity profiles the values of $\frac{\theta}{\delta}$, H_ϵ, H, l and D against K are shown in Table I (Choudhury 1967). With this system of profiles the compatibility condition (eqn. 7) takes the form,

$$\left(\frac{\theta}{\delta}\right)^2 (1+K) + \frac{\theta}{\delta}\lambda\left[\left(1-\frac{\pi}{6}\right)K\right] - \Lambda = 0. \quad \ldots \quad \ldots \quad (8)$$

TABLE I

Boundary layer characteristics for various values of the parameter K of Schlichting's profile

K	$\frac{\theta}{\delta}$	H_ϵ	H	l	D
+0·0	0·500	1·667	2·000	0·500	0·260
−0·1	0·493	1·657	2·046	0·470	0·239
−0·2	0·486	1·647	2·096	0·440	0·229
−0·3	0·478	1·637	2·149	0·410	0·219
−0·4	0·470	1·627	2·206	0·380	0·210
−0·5	0·461	1·616	2·268	0·351	0·202
−0·6	0·451	1·604	2·335	0·322	0·197
−0·7	0·442	1·592	2·406	0·294	0·187
−0·8	0·432	1·580	2·484	0·267	0·180
−0·9	0·421	1·567	2·568	0·241	0·174
−1·0	0·410	1·553	2·660	0·215	0·168
−1·1	0·398	1·539	2·761	0·189	0·163
−1·2	0·386	1·524	2·870	0·165	0·158
−1·3	0·373	1·509	2·991	0·142	0·153
−1·4	0·360	1·493	3·124	0·120	0·149
−1·5	0·347	1·475	3·271	0·099	0·145
−1·6	0·333	1·456	3·436	0·079	0·141
−1·7	0·319	1·436	3·619	0·061	0·137
−1·8	0·304	1·414	3·828	0·043	0·133
−1·9	0·288	1·390	4·063	0·027	0·129
−2·0	2·272	1·264	4·333	0·013	0·125
−2·099 Separation	0·256	1·336	4·642	0·000	0·120

6. Axisymmetric Stagnation Point Flow with Uniform Suction

(a) *Equations*

An investigation is being made into the axisymmetric stagnation point flow with uniform suction. A fluid stream impinges on a porous wall at right angles to it and flows away radially in all directions. A small uniform suction is imposed through the permeable surface of the body. The velocity for potential flow is given by

$$U(x) = \frac{U_0 x}{a}$$

where a is the representative length and $r(x) = x$ defines the body contour. Let

$$\bar{x} = \frac{x}{a}$$

$$\bar{r} = \frac{r(x)}{a} = \bar{x}$$

and

$$\bar{U} = \frac{U(x)}{U_0} = \bar{x}.$$

Then

$$\Lambda = t^* \frac{d\bar{U}}{d\bar{x}} = t^*$$

$$Z = t^* \frac{\bar{U}}{\bar{r}} \frac{d\bar{r}}{d\bar{x}} = t^*$$

$$\lambda = t^{*\frac{1}{2}} \bar{v}_s.$$

Thus for axisymmetric stagnation point flow along a porous wall the momentum integral eqn. (4), the kinetic energy integral eqn. (6) and the compatibility condition (7) become

$$\frac{dt^*}{d\bar{x}} = \frac{2}{\bar{x}} [l - (3+H)t^* + \bar{v}_s t^{*\frac{1}{2}}] \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

$$\frac{dH_\epsilon}{d\bar{x}} = \frac{1}{\bar{x}t^*} [2D - H_\epsilon \{l - (H-1)t^* + \bar{v}_s t^{*\frac{1}{2}}\} + \bar{v}_s t^{*\frac{1}{2}}] \quad \ldots \quad \ldots \quad (10)$$

and

$$\left(\frac{\theta}{\delta}\right)^2 (1+K) + \frac{\theta}{\delta} \bar{v}_s t^{*\frac{1}{2}} \left\{1 + \left(1 - \frac{\pi}{6}\right) K\right\} - t^* = 0. \quad \ldots \quad \ldots \quad (11)$$

At the stagnation point ($\bar{x} = 0$) the momentum integral and the kinetic energy integral equations exhibit singularity. If $\frac{dt^*}{d\bar{x}}$ and $\frac{dH_\epsilon}{d\bar{x}}$ are to remain finite at the stagnation point

$$l - (3+H)t^* + \bar{v}_s t^{*\frac{1}{2}} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

and

$$2D - H_\epsilon \{l - (H-1)t^* + \bar{v}_s t^{*\frac{1}{2}}\} + \bar{v}_s t^{*\frac{1}{2}} = 0. \quad \ldots \quad \ldots \quad (13)$$

These two equations together with the compatibility condition (eqn. 11) may be used to calculate the boundary layer for axisymmetric stagnation point flow along a porous wall.

(b) *Solution of the Momentum Integral Equation and the Compatibility Condition*

For impermeable wall, $\bar{v}_s = 0$, and eqns. (11) and (12) reduce to

$$\left(\frac{\theta}{\delta}\right)^2 (1+K) - t^* = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and

$$l-(3+H)t^* = 0. \quad \ldots \quad (15)$$

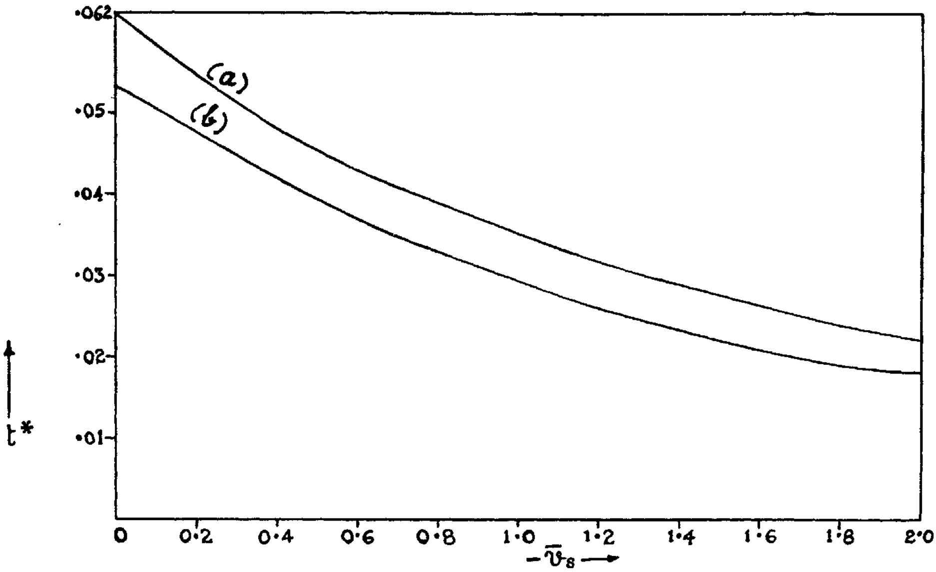

FIG. 1. Curves of t^* against $-\bar{v}_s$.

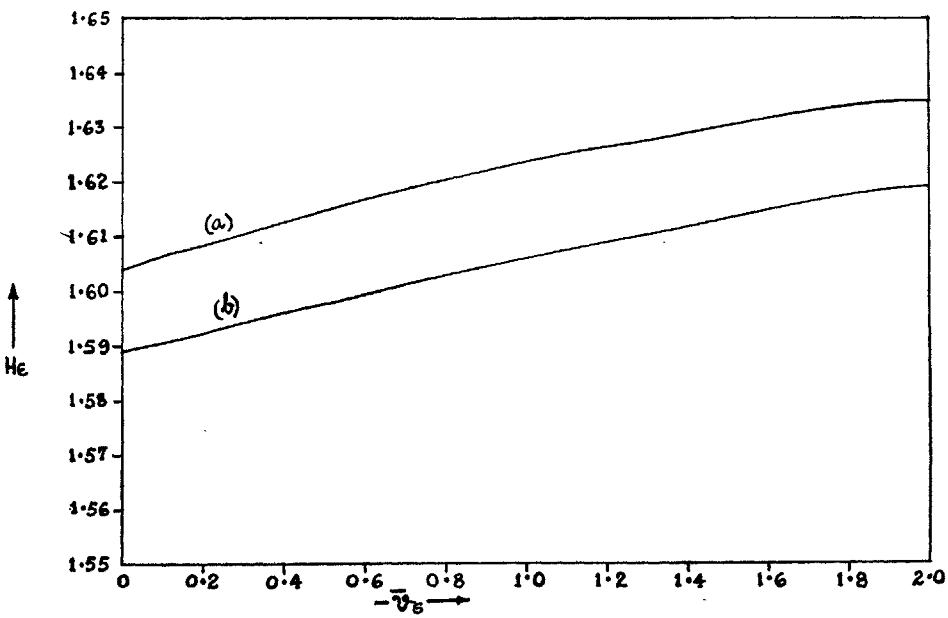

FIG. 2. Curves of H_ϵ against $-\bar{v}_s$.

The values obtained by the numerical solution of eqns. (14) and (15) with the help of Table I are:

$$t^* = 0{\cdot}0528$$
$$K = -0{\cdot}727$$
$$\frac{\theta}{\delta} = 0{\cdot}439$$
$$l = 0{\cdot}287$$
$$H = 2{\cdot}427.$$

For permeable surface $\bar{v}_s \neq 0$ and equations (11) and (12) have been solved numerically.

The variation of t^* and H_ϵ against the suction parameter $-\bar{v}_s$ is shown in Figs. 1 and 2 (*see* curve *b*).

(c) *Solution of the Momentum Integral and the Kinetic Energy Integral Equations*

More accurate results would be obtained by solving the momentum integral equation (12) and the kinetic energy integral equation (13).

For $\bar{v}_s = 0$, eqns. (12) and (13) reduce to

$$l-(3+H)t^* = 0 \qquad \ldots \ldots \ldots \ldots \quad (16)$$

and

$$2D-H_\epsilon\{l-(H-1)t^*\} = 0. \qquad \ldots \ldots \ldots \quad (17)$$

The values obtained by the numerical solution of eqns. (16) and (17) are:

$$t^* = 0{\cdot}0617$$
$$K = -0{\cdot}598$$
$$l = 0{\cdot}323$$
$$H = 2{\cdot}233$$
$$H_\epsilon = 1{\cdot}604$$
$$D = 0{\cdot}194.$$

For $\bar{v}_s \neq 0$, a particular value is assigned to K and the corresponding values of l, H, H_ϵ and D are read from Table I. With these values eqns. (12) and (13) are solved for $\bar{v}_s$ and t^*.

The variation of t^* and H_ϵ obtained by the solution of eqns. (12) and (13) is represented by curve (a) in Figs. 1 and 2.

A comparison of the values of t^* against the suction parameter $\bar{v}_s$ obtained by the various methods has been made in Table II.

For a solid wall problem ($\bar{v}_s = 0$) the value of the momentum thickness parameter t^* obtained by the present method of solution of the momentum integral and the kinetic energy integral equations is 0·0617 and that obtained by the solution of the momentum integral equation and the compatibility condition is 0·0528. The value given by Schlichting (1948) by the use of momentum equation is 0·0530 and that obtained by Truckenbrodt (1956) is

0·0491. According to the quadrature formula of Rott and Crabtree (1952) $t^* = 0{\cdot}0587$. The exact value given by Homann (1936) is 0·0610.

TABLE II

Comparison of the values of t^* for various values of $\bar{v}_s$

$\bar{v}_s$	t^*		
	Truckenbrodt (1956)	Solution of the momentum integral equation and the compatibility condition	Present method of solution of the momentum integral and the kinetic energy integral equations
0·0	0·049	0·053	0·062
−0·2	0·048	0·047	0·054
−0·4	0·044	0·041	0·048
−0·6	0·041	0·036	0·043
−0·8	0·037	0·033	0·039
−1·0	0·033	0·029	0·035
−1·2	0·031	0·026	0·032
−1·4	0·028	0·024	0·029
−1·6	0·026	0·021	0·026
−1·8	0·024	0·019	0·024
−2·0	0·022	0·018	0·022

The above comparison of the values of t^* for the axisymmetric stagnation point flow along a solid wall shows that the result obtained by the use of the momentum integral and the kinetic energy integral equations is the most satisfactory and is closest to the exact value of Homann (1936). Therefore, the value of t^* and the other parameters for various values of $\bar{v}_s$ obtained by the solution of the momentum integral and the kinetic energy integral equations [curve (*a*) in Figs. 1 and 2] can be taken as representing the most accurate values.

$$\frac{dt^*}{d\bar{x}} \text{ and } \frac{dH_\epsilon}{d\bar{x}}$$

are indeterminate at the stagnation point. Their values are obtained by going over to the limit with the aid of the compatibility condition. For uniform suction both

$$\frac{dt^*}{d\bar{x}} \text{ and } \frac{dH_\epsilon}{d\bar{x}}$$

are seen to vanish at the stagnation point with the result that t^* and H_ϵ remain independent of $\bar{x}$ near the stagnation point.

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