SHOCK WAVES THROUGH SELF-GRAVITATING GAS SPHERES

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The present investigation deals with the propagation of strong shock waves through the envelope of a generalized Roche model. By using the method of characteristics analytic expressions for shock velocity and flow variables behind the shock are obtained. These expressions show that the effects of density distribution (density proportional to the square of the radial distance from the centre) and self-gravitating force are of opposite nature. Due to this density distribution, an increase in radial distance leads to an increase in shock velocity, pressure and particle velocity; and due to the self-gravitating force, an increase in the radial distance leads to a decrease in shock velocity, pressure and particle velocity. Finally, it is concluded that for this model these expressions are such functions of radial distance r that the effects due to self-gravitating force starts dominating for large values of r; and as a result, ultimately shock decays.

Introduction

A study of propagation of spherical shock waves through self-gravitating gas spheres—Stellar models—is of importance from astrophysical point of view (Nova outbursts etc.). Carrus *et al.* (1951*a*) studied the shock wave propagation through the envelope of the generalized Roche model. This expanding shock wave is produced by the expansion of the core which in turn is a consequence of a sufficiently strong instantaneous central explosion. They found that for γ, adiabatic index of envelope gas, $= \frac{4}{3}$, the core will be of zero radius. It may be emphasized that in this model there existed a discontinuity in density across the interface of the core and its envelope. In their second paper Carrus *et al.* (1951*b*) investigated the shock wave propagation through self-gravitating gas spheres with density as a continuous function of radial distance. In this case they found that central part of the configuration will be effectively evacuated by the central explosion. Kopal (1954) studied a new aspect of explosion, i.e. the energy released in the explosions is very small in comparison to the energy of the gas sphere in its pre-explosion state. Here it may be noticed that the energy of the shocked gas is increasing with time; whereas in the previous two papers by Carrus *et al.* (1951*a*, *b*) energy was considered constant with respect to time. Similarity method was adopted for solving the above problems. Sakurai (1956), using similarity method and perturbation technique, attempted to study the propagation of strong shock waves of varying strength (in the above-mentioned three papers shock strength was constant) through self-gravitating gas spheres. He expanded all the non-dimensional flow variables in power series of $\left(\frac{R}{R_c}\right)$, where R is shock radius and R_c is a characteristic radius, and studied the successive approximations. The results of Sakurai's investigation are of value in the study of Super nova phenomenon; but this series fails to converge for the greater part of the star, in case of Nova. The shock after travelling about 10 per cent or less of Stellar radius will become sufficiently weak and the equations can be linearized. This linearized problem was discussed in detail by Whitham (1953). Rogers (1956, 1957) studied the propagation of shock waves of varying strength through self-gravitating gas spheres and observed that density distribution ahead of shock has an important effect on the shock strength variation.

Attempts have been made to study the effects of density distribution (Whitham 1958) or the combined effects of density distribution and self-gravitational force (Kopal 1954, Rogers 1956, Sakurai 1956, etc.) on shock wave propagation; but the individual effect of self-gravitational force which interacts with inertia forces has not been studied so far. To study these effects is the aim of the present investigation. However, the effects due to density distribution are also studied.

Now let us consider the propagation of strong shock waves through the envelope of generalized Roche model, as considered by Carrus *et al.* (1951*a*). Shock is supposed to be sufficiently strong, so that strong shock conditions can be used for Rankine-Hugoniot conditions. By following the method of characteristics, adopted by Whitham, analytic expressions are obtained for shock velocity and flow variables behind the shock. The expressions for shock velocity, pressure and particle velocity consist of two terms: One due to density distribution and the other due to self-gravitating force. It is interesting to note that these two terms have opposite effects on shock wave propagation.

Basic Equations and Solutions

If m be mass of massive core which is constant, $-\frac{Gm}{r^2}$ will be the gravitational force at radial distance r from the centre of sphere, then the momentum equation will be

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{\partial p}{\partial r}+\frac{Gm}{r^2}=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where ρ, u, p are respectively density, velocity and pressure. G is gravitational constant. The equations of conservation of mass and entropy for a polytropic gas will be

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial t}+\rho\frac{\partial u}{\partial r}+\frac{2u\rho}{r}=0 \quad \ldots \quad (2)$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial r}+a^2\rho\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)=0 \quad \ldots \quad (3)$$

where a is the local velocity of sound. By multiplying eqn. (1) by a and adding to (3), we get

$$\frac{\partial p}{\partial t}+(u+a)\frac{\partial p}{\partial r}+\rho a\left\{\frac{\partial u}{\partial t}+(u+a)\frac{\partial u}{\partial r}\right\}+\frac{Gma}{r^2}+\frac{2a^2\rho u}{r}=0. \quad \ldots \quad (4)$$

This equation along a characteristic $\left(\frac{dr}{dt}=u+a\right)$ can be written as

$$dp+\rho a\,du+\frac{Gma\,dr}{r^2(u+a)}+\frac{2a^2\rho u\,dr}{r(u+a)}=0.$$

For a strong shock, boundary conditions for eqns. (1) to (3) are

$$\rho=\frac{\gamma+1}{\gamma-1}\rho_0,\quad p=\frac{2\rho_0}{\gamma+1}U^2,\quad u=\frac{2}{\gamma+1}U,\quad a^2=\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}U^2 \quad \ldots \quad (5)$$

where U is shock velocity and subscript zero stands for the state ahead of shock. Now eqn. (4) with the help of (5) can be written in the following form:

$$\frac{dU^2}{dr}+\left\{\frac{2}{A\rho_0}\frac{d\rho_0}{dr}+\frac{8r}{rAB}\right\}U^2=-\frac{GmC}{Ar^2\rho_0} \quad \ldots \quad (6)$$

where

$$A=2+\frac{\sqrt{2\gamma(\gamma-1)}}{\gamma-1},\quad B=2+\sqrt{2\gamma(\gamma-1)},\quad C=\sqrt{2\gamma(\gamma-1)}.$$

The eqn. (6) is an ordinary linear differential equation of first order in U^2; therefore, its solution can be easily obtained and written as

$$U^2=\rho_0^{-\frac{2}{A}}r^{-\frac{8\gamma}{AB}}\left\{K-\int\frac{GmC}{Ar^2\rho_0}\rho_0^{\frac{2}{A}}r^{\frac{8\gamma}{AB}}dr\right\} \quad \ldots \quad (7)$$

where K is an arbitrary constant.

Discussions and Conclusions

The expression for shock velocity, given by (7), clearly indicates the effects of any given density distribution and self-gravitating force. However, to get more insight into the phenomenon and to know more about this function a special case has been considered. We consider

$$\rho_0=\beta r^{-2} \quad \ldots \quad (8)$$

where β is a constant. With this density distribution eqn. (7) reduces to

$$U^2 = K\beta^{-\frac{2}{A}} r^{\frac{4}{A}-\frac{8\gamma}{AB}} - \frac{GmC}{\beta} \frac{r}{\frac{8\gamma}{B}-4+A}. \qquad (9)$$

From the form of the first term of eqn. (9), which is due to density distribution, it is evident that any expanding shock propagating into a medium with density proportional to the inverse square of the radial distance will strengthen as the distance of the shock front from the centre increases (because power of r is positive). This is in agreement with the result obtained by Whitham (1958). The second term of (9), which is due to self-gravitating force, shows that shock weakens as it propagates away from the centre. Thus, we see that in this model the effects of density distribution and self-gravitating force are of opposite nature. The eqn. (9) considerably simplifies for $\gamma = 2$ and is given by

$$U^2 = \frac{k}{\sqrt{\beta}} - \frac{Gmr}{2\beta}. \qquad (10)$$

The expressions for pressure and particle velocity can be obtained without any difficulty from eqns. (5) and (9). It may also be noticed from (9) that as r starts increasing the value of U^2 increases and reaches a maximum value and then starts decreasing. Whether this type of behaviour of shock velocity will be observed in the envelope or not depends on the value of constants γ, β, k, m and G. The maximum value of U will be at distance $r_{\max}$ which is given by

$$r_{\max} = \left\{ \frac{GmC\beta^{\frac{2}{A}-1}}{k\left(\frac{4}{A}-\frac{8\gamma}{AB}\right)\left(\frac{8\gamma}{B}-4+A\right)} \right\}^{\frac{1}{\left(\frac{4}{A}-\frac{8\gamma}{AB}-1\right)}}. \qquad (11)$$

At this stage, it may also be added that as the radial distance r increases the right-hand side of (9) will become negative and for these values of r the present solution will not be valid. Perhaps, before this situation arises shock may be reflected and the solution will be given by negative characteristics instead of positive characteristics. Further, it is observed from (9) that the value of shock velocity is governed by the value of adiabatic index. This is in contrast to the work of Carrus *et al.* (1951*a*), where they have found shock velocity independent of the value of adiabatic index.

The author also finds it necessary to compare the results of the present investigation to those of Sakurai's (1956) results. According to Sakurai, the net effect of density distribution and self-gravitational force is 'decay of shock' (*see* Sakurai 1956, Fig. 3); whereas in the present case we see that net result is: at first, shock velocity starts increasing and reaches its maximum value and then starts decreasing. Secondly, it is evident (*see* Sakurai 1956, Fig. 3) that an increase in the value of G leads to an increase in the value of shock velocity; in the present case (from (9)), an increase in the value of G leads to a decrease in the value of shock velocity. No doubt, the present model and Sakurai's model are different but still some agreement of results can be expected because basically both problems are the same. To the author this disagreement is little short of surprise and the reasons for this disagreement are far from evident. In conclusion it may be said that in spite of the above disagreements, the main result of the present investigation agrees with Sakurai's result because the power of first term of (9) is always positive and less than one, therefore the effects of second term will be dominating for large values of r; as a result, finally, the shock will decay, which is in agreement with Sakurai's result (1956, Fig. 3).

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