ON THE FLOW OF RIVLIN-ERICKSEN FLUIDS IN A CHANNEL BOUNDED BY TWO PARALLEL FLAT PLATES

by S. N. DŪBE,* *Department of Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 5*

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In the present note the flow of Rivlin-Ericksen fluids in a channel bounded by two parallel flat plates under the influence of axial pressure gradient (i) varying linearly with time and (ii) decreasing exponentially with time is considered. Effect of elastic parameter of the fluid on their behaviour is sought.

INTRODUCTION

The present note consists of two parts. In part I the flow in a channel bounded by two parallel flat plates under axial pressure gradient varying linearly with time is discussed. This consists of two parts, the one varies linearly with the parameter $T = \frac{\alpha t}{{y_0}^2}$ and the other is transient part of the velocity, which vanishes in the limit as t tends to infinity. It is also seen that the contribution of the transient part is insignificant for very large T ($\gg 1$).

In part II the flow in the same channel under exponentially decreasing pressure gradient is studied. An expression for the velocity has been obtained taking

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = a_0 + \sum_{m=1}^{\infty} a_m e^{-mt},$$

and the expression obtained has been compared with that of Rajeswari's result (1963). Our expression contains some additional terms and the reason for this has been discussed.

1. FORMULATION OF THE PROBLEM

The constitutive equation for a viscoelastic fluid as given by Rivlin and Ericksen (1955) has the following form:

$$T_{ij} = -P\delta_{ij} + P_{ij} \qquad \ldots \quad (1.1)$$

$$P_{ij} = \phi_1 E_{ij} + \phi_2 D_{ij} + \phi_3 E_{im} E_{mj} \qquad \ldots \quad (1.2)$$

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* Associate Professor, Department of Mathematics, Centre for P.G. Studies, Himachal Pradesh University, Simla.

$$E_{ij} = U_{i,j}+U_{j,i} \quad \ldots \quad (1.3)$$

$$D_{ij} = A_{i,j}+A_{j,i}+2U_{m,i}U_{m,j} \quad \ldots \quad (1.4)$$

$$A_i = \frac{\partial U_i}{\partial t}+U_m U_{i,m} \quad \ldots \quad (1.5)$$

where the symbols have their usual meanings and a suffix following a comma denotes covariant differentiation.

The equations of motion and continuity have the following forms:

$$\rho A_i = T_{ij,j} \quad \ldots \quad (1.6)$$

$$U_{i,i} = 0 \quad \ldots \quad (1.7)$$

where ρ is the density of the fluid.

For the present problem we have

$$\left.\begin{array}{ll} u = u(x, y, t), & v = 0, w = 0 \\ P = P(x, y, t), & \frac{\partial}{\partial z}(\quad) = 0 \end{array}\right\} \quad \ldots \quad (1.8)$$

where u, v, w are the physical components of the velocity vector. The last equation of (1.8) holds because the motion is two-dimensional. Furthermore, the equation of continuity (1.7) and the conditions (1.8) give

$$\frac{\partial u}{\partial x} = 0 \text{ so that } u = u(y, t). \quad \ldots \quad (1.9)$$

Substituting eqns. (1.8) and (1.9) into the equations of motion, we get

$$\frac{1}{\rho}\frac{\partial P}{\partial y} = (2\beta+\gamma)\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right] \quad \ldots \quad (1.10)$$

and

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x}+\alpha\frac{\partial^2 u}{\partial y^2}+\beta\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right) \quad \ldots \quad (1.11)$$

where $\alpha = \frac{\phi_1}{\rho}$, $\beta = \frac{\phi_2}{\rho}$ and $\gamma = \frac{\phi_3}{\rho}$ are the kinematical coefficients of viscosity, viscoelasticity and cross-viscosity respectively.

Equation (1.11) determines the velocity field $u(y, t)$, which shows that the coefficient of cross-viscosity does not affect the velocity field, as in all two-dimensional flows, but modifies the pressure field. Equation (1.10) gives the variation of the pressure field across the channel, which will not be discussed further in the present investigation.

PART I

2. Axial Pressure Gradient varies Linearly with Time

Let us assume that

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = a_0 + at. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

Equation (1.11) then becomes

$$\frac{\partial u}{\partial t} = a_0 + at + \alpha\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right). \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

Let $\bar{u} = \int_0^\infty e^{-st} u \, dt$ be the Laplace transform of u and let u_0 be the initial value of u.

Multiplying eqn. (2.2) by e^{-st} and then integrating between the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{u}}{\partial y^2} - p^2 \bar{u} = -\frac{1}{(\alpha+\beta s)}\left[u_0 - \beta\frac{\partial^2 u_0}{\partial y^2} + \frac{a_0}{s} + \frac{a}{s^2}\right] \quad \ldots \quad \ldots \quad (2.3)$$

where $p^2 = \dfrac{s}{(\alpha+\beta s)}$.

We shall now find u_0.

Initially the pressure gradient is a_0 and the motion is steady in the channel. Hence

$$\frac{d^2 u_0}{dy^2} = -\frac{a_0}{\alpha}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

The boundary conditions are

$$u_0 = 0 \text{ when } y = -y_0$$

$$\text{and } u_0 = 0 \text{ when } y = y_0.$$

The solution of (2.4) with the above boundary conditions is

$$u_0 = \frac{a_0}{2\alpha}(y_0^2 - y^2).$$

Substituting this value of u_0 in (2.3), we get

$$\frac{\partial^2 \bar{u}}{\partial y^2} - p^2 \bar{u} = -\frac{1}{(\alpha+\beta s)}\left[\frac{\beta a_0}{\alpha} + \frac{a_0}{2\alpha}(y_0^2 - y^2) + \frac{a_0}{s} + \frac{a}{s^2}\right]. \quad \ldots \quad (2.5)$$

The boundary conditions for $\bar{u}$ are

$$\bar{u} = 0 \text{ when } y = -y_0$$

$$\text{and } \bar{u} = 0 \text{ when } y = y_0.$$

The solution of eqn. (2.5) under these boundary conditions is

$$\bar{u} = \frac{a_0}{2\alpha}\left(\frac{y_0^2 - y^2}{s}\right) + \frac{a}{s^2}\left[1 - \frac{\cosh py}{\cosh py_0}\right].$$

Now applying Laplace inversion theorem, we get

$$u = \frac{a_0}{2\alpha}(y_0^2-y^2)+\frac{1}{2\alpha}(y_0^2-y^2)at-\frac{a}{24\alpha^2}(5y_0^2-y^2)(y_0^2-y^2)-\frac{a\beta}{2\alpha^2}(y_0^2-y^2)$$

$$+\frac{64ay_0^4}{\alpha^2\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n\left[1+\frac{\beta(2n+1)^2\pi^2}{4y_0^2}\right]}{(2n+1)^5}\cdot\exp\left\{-\frac{\frac{\alpha(2n+1)^2\pi^2}{4y_0^2}}{1+\beta\frac{(2n+1)^2\pi^2}{4y_0^2}}t\right\}\cdot\cos\left\{\frac{(2n+1)\pi y}{2y_0}\right\}. \quad (2.6)$$

We now make eqn. (2.6) dimensionless by introducing

$$U = \frac{u}{U_0}, \quad \frac{y}{y_0} = r, \quad T = \frac{\alpha t}{y_0^2}$$

where U_0 represents a characteristic velocity, for example, velocity on the axis of the channel.

We then get

$$U = b_0(1-r^2)+bT(1-r^2)-\frac{b}{12}(5-r^2)(1-r^2)-bb_1(1-r^2)$$

$$+\frac{128b}{\pi^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^5}\left[1+b_1\frac{(2n+1)^2\pi^2}{4}\right]\cdot\exp\left\{\frac{\frac{(2n+1)^2\pi^2}{4}}{1+b_1\frac{(2n+1)^2\pi^2}{4}}T\right\}\cdot\cos\left\{\frac{(2n+1)\pi r}{2}\right\} \quad (2.7)$$

where

$$b_0 = \frac{a_0y_0^2}{2\alpha U_0}, \text{ a non-dimensional number,}$$

$$b = \frac{ay_0^4}{2\alpha^2 U_0}, \text{ a non-dimensional number,}$$

$$b_1 = \frac{\beta}{y_0^2}, \text{ the elastic number.}$$

We note the following important points:

(i) The elastic number b_1 of the liquid reduces the speed of flow.

(ii) In the Newtonian case it has been shown by Dube (1969) that the transient part becomes insignificant for $T > 1$, but in the present case it is seen from the expression (2.7) that the transient part will be insignificant for very large T ($\gg 1$). It means that Rivlin-Ericksen fluids' velocity will take more time to vary linearly with T than the velocity of Newtonian fluid. This difference, of course, is due to the presence of elasticity in the fluid.

(iii) From the expression (2.7) it is observed that U increases with T for fixed r. It is also seen that for any value of T, U decreases with the increase of r and it is maximum when $r = 0$. It means that the points near the axis of the channel move faster than the points which are far from the axis of the channel.

(iv) We also observe from the expression (2.7) that U will not change whether r is negative or positive. It means that the symmetrical points have the same velocity.

PART II

3. Axial Pressure Gradient decreases Exponentially with Time

We now take

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = a_0 + \sum_{m=1}^{\infty} a_m e^{-mt}. \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Equation (1.11) then becomes

$$\frac{\partial u}{\partial t} = a_0 + \sum_{m=1}^{\infty} a_m e^{-mt} + \alpha\frac{\partial^2 u}{\partial y^2} + \beta\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right). \qquad \ldots \quad \ldots \quad (3.2)$$

Let $\bar{u} = \int_0^\infty e^{-st} u\, dt$ be the Laplace transform of u and let u_0 be the initial value of u.

Multiplying eqn. (3.2) by e^{-st} and integrating between the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{u}}{\partial y^2} - p^2\bar{u} = -\frac{1}{(\alpha+\beta s)}\left[u_0 - \beta\frac{\partial^2 u_0}{\partial y^2} + \frac{a_0}{s} + \sum_{m=1}^{\infty}\frac{a_m}{s+m}\right] \qquad \ldots \quad (3.3)$$

where $p^2 = \dfrac{s}{(\alpha+\beta s)}$.

Here again

$$u_0 = \frac{a_0}{2\alpha}(y_0^2 - y^2).$$

The solution of (3.3) under the boundary conditions

$$\bar{u} = 0 \text{ when } y = -y_0$$
$$\text{and } \bar{u} = 0 \text{ when } y = y_0$$

is

$$\bar{u} = \frac{a_0}{2\alpha}\left(\frac{y_0^2 - y^2}{s}\right) + \left[1 - \frac{\cosh py}{\cosh py_0}\right]\sum_{m=1}^{\infty}\frac{a_m}{s(s+m)}.$$

Now applying Laplace inversion theorem, we get

$$u = \frac{a_0}{2\alpha}(y_0^2 - y^2) - \sum_{m=1}^{\infty}\frac{a_m}{m}\left[1 - \frac{\cos\left\{\left(\dfrac{m}{\alpha-\beta m}\right)^{\frac{1}{2}} y\right\}}{\cos\left\{\left(\dfrac{m}{\alpha-\beta m}\right)^{\frac{1}{2}} y_0\right\}}\right] e^{-mt}$$
$$+ \frac{4}{\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a_m}{(2n+1)(m+\lambda)\left[1+\dfrac{\beta(2n+1)^2\pi^2}{4y_0^2}\right]}\cdot e^{\lambda t}\cdot\cos\left\{\frac{(2n+1)\pi y}{2y_0}\right\}$$

where

$$\lambda = -\frac{\dfrac{\alpha(2n+1)^2\pi^2}{4y_0^{\,2}}}{1+\beta\dfrac{(2n+1)^2\pi^2}{4y_0^{\,2}}}$$

or

$$u = u'_1+u'_2+u'_3. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

This expression for the velocity does not agree with Rajeswari's result (1963). Her expression does not contain u'_3. This difference lies in the fact that Rajeswari has assumed the form of u as

$$u = u_0+\sum_{m=1}^{\infty} u_m e^{-mt}$$

where u_0 and u_m are functions of y only. Naturally then the part u'_3 will be absent in her expression. But $u'_1+u'_2+u'_3$ is a more general solution of (3.2) and this is confirmed by Laplace transform method used in the present note.

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