VISCOUS INCOMPRESSIBLE STEADY LAMINAR FLOW BETWEEN TWO POROUS COAXIAL ROTATING CIRCULAR CYLINDERS WITH DIFFERENT PERMEABILITY

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The flow of viscous incompressible fluid between two porous coaxial rotating circular cylinders with different radial velocities v_a and v_b at the walls has been studied with the help of perturbation technique. An exact solution of the Navier-Stokes equations reduced to second and third order non-linear differential equations with appropriate boundary conditions has been obtained. The results, when the outer cylinder or the inner cylinder is rigid or both the cylinders have the same permeability, have been obtained as particular cases of the investigation. When the outer cylinder is rigid, the results correspond to the results of Gupta and Goyal (1972) and when the cylinders have the same permeability, the results can easily be compared with the results of Kapur and Malick (1960). In the case when the inner cylinder is rotating with twice the angular velocity of the outer cylinder and the permeability of the inner cylinder is greater than that of the outer cylinder, there is a decrease in the azimuthal velocity as well as in the axial pressure gradient. Also the maximum of the longitudinal velocity shifts towards the inner cylinder.

INTRODUCTION

In the last few years a number of persons have studied the flow of viscous incompressible fluid with various types of suction and injection. Berman (1953) for the first time considered the effect of wall porosity on velocity and pressure distribution for the channel of rectangular cross-section. With the assumption of uniform wall porosity, he obtained the required expressions by solving the differential equations with the perturbation technique. Afterwards Sellars (1955), Yuan (1956), Yuan and Finkelstein (1956), Berman (1958) and Terril and Shrestha (1965) also contributed for the flow through porous walls. The radial velocity component was produced by injecting the fluid through one wall and removing it through the other wall. Verma and Bansal (1969) studied the laminar flow in an annulus with porous walls of different permeability. Kapur and Malick (1960) and Sinha and Chaudhary (1966) considered the effect of wall porosity on velocity, pressure and temperature distribution with rotating annulus. Kapur and Malick (1960) obtained interesting results concerning points of inflexion as well as points of maxima and minima. Gupta and Goyal (1972) also studied the viscous incompressible flow between two coaxial rotating circular cylinders with small uniform injection at the inner cylinder. They got the expressions for all the three velocity profiles (u, v, w) with the help of perturbation technique.

In this paper we have studied the flow of viscous incompressible fluid between two porous coaxial rotating circular cylinders with different radial velocities v_a and v_b at the walls. With the help of perturbation technique, an exact solution of the Navier-Stokes equations reduced to second and third order non-linear differential equations with appropriate boundary conditions has been obtained.

The results, when the outer cylinder or the inner cylinder is rigid or both the cylinders have the same permeability, have been obtained as particular cases of the investigation. When the outer cylinder is rigid, the results correspond to the results of Gupta and Goyal (1972) and when the cylinders have the same permeability, the results can easily be compared with the results of Kapur and Malick (1960). In the case when the inner cylinder is rotating with twice the angular velocity of the outer cylinder and the permeability of the inner cylinder is greater than that of the outer cylinder, there is a decrease in the azimuthal velocity as well as in the axial pressure gradient. Also the maximum of the longitudinal velocity shifts towards the inner cylinder. It is interesting to note that

(i) The longitudinal and transverse velocities are independent of the rates of rotation of cylinders;

(ii) The radial and azimuthal components of velocity are the functions of radial distance η only: though they depend on σ the radius ratio of annulus and (λ_1, λ_2) the radial velocity parameters;

(iii) The results hold good for all the values of the suction and injection parameters $(\lambda_1 < 1, \lambda_2 < 1)$; and

(iv) The azimuthal velocity profiles continue to remain parabolic for all the ratios of the angular velocities of the cylinders.

The longitudinal, transversal and azimuthal velocity profiles have been drawn for some values of λ_1 and $\lambda_2[(0 \leqslant \lambda_1 < 1),\ (0 \leqslant \lambda_2 < 1),\ (\lambda_1+\lambda_2) < 1]$. The graphs for the longitudinal pressure distribution have been drawn for some values of λ_1 and λ_2. The graphs for all the above quantities have also been drawn in some particular cases, viz.

(i) when the inner cylinder is rigid, and

(ii) when both the cylinders have the same permeability.

Equations of Motion

Let us consider the steady laminar flow of a viscous incompressible fluid in a region bounded by two coaxial circular cylinders of radii a and b, rotating with constant angular velocities ω_1 and ω_2 respectively. It is assumed that the fluid is injected at the inner cylinder with velocity v_b and sucked at the outer cylinder with velocity v_a.

The Navier-Stokes equations of motion in cylindrical coordinates (r, θ, x) are

$$\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{w}{r}\frac{\partial v}{\partial \theta}+u\frac{\partial v}{\partial x}-\frac{w^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r}+\mu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial x^2}-\frac{v}{r^2}-\frac{2}{r^2}\frac{\partial w}{\partial \theta}\right] \quad .. \quad (1.1)$$

$$\rho\left(\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial r}+\frac{w}{r}\frac{\partial w}{\partial \theta}+u\frac{\partial w}{\partial x}+\frac{vw}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial x^2}+\frac{2}{r^2}\frac{\partial v}{\partial \theta}-\frac{w}{r^2}\right] \quad (1.2)$$

$$\rho\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r}+\frac{w}{r}\frac{\partial u}{\partial \theta}+u\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial x^2}\right] \quad .. \quad .. \quad (1.3)$$

and the equation of continuity is

$$\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial u}{\partial x}=0, \quad .. \quad .. \quad .. \quad (1.4)$$

where u, v, w represent the axial, radial and azimuthal components of velocity in the direction of x, r, θ respectively.
But

$\frac{\partial(\;)}{\partial t} = 0$ for steady motion,

$\frac{\partial(\;)}{\partial \theta} = 0$ for axial symmetry,

and

$\frac{\partial w}{\partial x} = 0$ for azimuthal velocity produced due to rotation only.

Hence (1.1) to (1.4) become

$$v\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial x^2}\right] \quad .. \quad .. \quad .. \quad (1.5)$$

$$v\frac{\partial v}{\partial r}+u\frac{\partial v}{\partial x}-\frac{w^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial x^2}-\frac{v}{r^2}\right] \quad .. \quad (1.6)$$

$$v\frac{\partial w}{\partial r}+\frac{vw}{r} = \nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{w}{r^2}\right] \quad .. \quad .. \quad .. \quad .. \quad (1.7)$$

and

$$\frac{\partial v}{\partial r}+\frac{v}{r}+\frac{\partial u}{\partial x}=0 \quad \ldots \quad (1.8)$$

the boundary conditions being:

$$\left.\begin{array}{l} r=a:\ u=0,\ v=v_a,\ w=a\omega_1 \\ r=b:\ u=0,\ v=v_b,\ w=b\omega_2 \end{array}\right\}. \quad \ldots \quad (1.9)$$

Now introducing the following non-dimensional quantities

$$\eta=\frac{r}{b},\ \bar{x}=\frac{x}{b},\ \bar{u}=\frac{ub}{\nu},\ \bar{v}=\frac{vb}{\nu},\ \bar{w}=\frac{wb}{\nu},$$

$$\bar{p}=\frac{pb^2}{\nu^2\rho},\ \lambda_1=\frac{av_a}{\nu},\ \lambda_2=\frac{bv_b}{\nu},\ \Omega_1=\frac{ab\omega_1}{\nu},\ \Omega_2=\frac{b^2\omega_2}{\nu},$$

and

$$\sigma=\frac{a}{b}$$

where

$$\sigma \geqslant \eta \geqslant 1, \quad \ldots \quad (1.10)$$

eqns. (1.5) to (1.8) become,

$$\bar{v}\frac{\partial \bar{u}}{\partial \eta}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial^2 \bar{u}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{u}}{\partial \eta}+\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \quad \ldots \quad (1.11)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \eta}+\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}-\frac{\bar{w}^2}{\eta}=-\frac{\partial \bar{p}}{\partial \eta}+\frac{\partial^2 \bar{v}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{v}}{\partial \eta}+\frac{\partial^2 \bar{v}}{\partial \bar{x}^2}-\frac{\bar{v}}{\eta^2} \quad \ldots \quad (1.12)$$

$$\bar{v}\frac{\partial \bar{w}}{\partial \eta}+\frac{\bar{v}\bar{w}}{\eta}=\frac{\partial^2 \bar{w}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{w}}{\partial \eta}-\frac{\bar{w}}{\eta^2} \quad \ldots \quad (1.13)$$

and

$$\frac{\partial \bar{v}}{\partial \eta}+\frac{\bar{v}}{\eta}+\frac{\partial \bar{u}}{\partial \bar{x}}=0. \quad \ldots \quad (1.14)$$

From eqn. (1.13) it is evident that $\bar{v}$ cannot be a function of $\bar{x}$, hence

$$\frac{\partial \bar{v}}{\partial \bar{x}}=0 \quad \ldots \quad (1.15)$$

and from (1.14), we get

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}=0. \quad \ldots \quad (1.16)$$

Applying (1.15) and (1.16) in eqns. (1.11) to (1.14), we have

$$\bar{v}\frac{\partial \bar{u}}{\partial \eta}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial^2 \bar{u}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{u}}{\partial \eta} \quad \ldots \quad (1.17)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \eta}-\frac{\bar{w}^2}{\eta}=-\frac{\partial \bar{p}}{\partial \eta}+\frac{\partial^2 \bar{v}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{v}}{\partial \eta}-\frac{\bar{v}}{\eta^2} \quad \ldots \quad (1.18)$$

$$\bar{v}\frac{d\bar{w}}{d\eta}+\frac{\bar{v}\bar{w}}{\eta}=\frac{d^2\bar{w}}{d\eta^2}+\frac{1}{\eta}\frac{d\bar{w}}{d\eta}-\frac{\bar{w}}{\eta^2} \quad \ldots \quad (1.19)$$

and

$$\frac{\partial \bar{v}}{\partial \eta}+\frac{\bar{v}}{\eta}+\frac{\partial \bar{u}}{\partial \bar{x}}=0 \qquad \text{(1.20)}$$

with the boundary conditions

$$\left.\begin{aligned} \eta=\sigma:\ \bar{u}=0,\ \bar{v}=\frac{\lambda_1}{\sigma},\ \bar{w}=\Omega_1 \\ \eta=1:\ \bar{u}=0,\ \bar{v}=\lambda_2,\ \bar{w}=\Omega_2. \end{aligned}\right\} \qquad \text{(1.21)}$$

Let

$$\left.\begin{aligned} \bar{p} &= p_0(\eta)+p'(\bar{x},\eta) \\ \bar{u} &= u'(\bar{x},\eta) \\ \bar{v} &= v'(\eta) \\ \bar{w} &= w_0(\eta)+w'(\eta) \end{aligned}\right\} \qquad \text{(1.22)}$$

where the primed quantities are the perturbations caused by the suction and p_0 and w_0 are the known quantities for the non-porous walls, satisfying the equations (Shih-I-Pai 1956):

$$\frac{\partial p_0}{\partial \bar{x}}=0 \qquad \text{(1.23)}$$

$$\frac{w_0^2}{\eta}=\frac{\partial p_0}{\partial \eta} \qquad \text{(1.24)}$$

and

$$\frac{d^2 w_0}{d\eta^2}+\frac{1}{\eta}\frac{dw_0}{d\eta}-\frac{w_0}{\eta^2}=0 \qquad \text{(1.25)}$$

with the boundary conditions

$$\eta=\sigma:\ w_0=\Omega_1;\ \eta=1:\ w_0=\Omega_2. \qquad \text{(1.26)}$$

From (1.25) and (1.26), we have

$$w_0=\frac{1}{1-\sigma^2}\left[(\Omega_2-\Omega_1\sigma)\eta+\frac{\sigma}{\eta}(\Omega_1-\Omega_2\sigma)\right]. \qquad \text{(1.27)}$$

Now using (1.22) to (1.25), eqns. (1.17) to (1.20) become

$$v'\frac{\partial u'}{\partial \eta}+u'\frac{\partial u'}{\partial \bar{x}}=-\frac{\partial p'}{\partial \bar{x}}+\frac{\partial^2 u'}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u'}{\partial \eta} \qquad \text{(1.28)}$$

$$v'\frac{\partial v'}{\partial \eta}-\frac{2w_0w'}{\eta}-\frac{w'^2}{\eta}=-\frac{\partial p'}{\partial \eta}+\frac{\partial^2 v'}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial v'}{\partial \eta}-\frac{v'}{\eta^2} \qquad \text{(1.29)}$$

$$v'\left(\frac{dw_0}{d\eta}+\frac{dw'}{d\eta}\right)+\frac{v'}{\eta}(w_0+w')=\frac{d^2w'}{d\eta^2}+\frac{1}{\eta}\frac{dw'}{d\eta}-\frac{w'}{\eta^2} \qquad \text{(1.30)}$$

and

$$\frac{\partial v'}{\partial \eta}+\frac{v'}{\eta}+\frac{\partial u'}{\partial \bar{x}}=0 \qquad \text{(1.31)}$$

with the boundary conditions

$$\left.\begin{aligned} \eta=\sigma:\ u'=0,\ v'=\frac{\lambda_1}{\sigma},\ w'=0 \\ \eta=1:\ u'=0,\ v'=\lambda_2,\ w'=0 \end{aligned}\right\}. \qquad \text{(1.32)}$$

Method of Solution

Let

$$v' = \frac{f(\eta)}{\eta}\lambda_1 + \frac{\phi(\eta)}{\eta}\lambda_2. \quad \ldots \quad (2.1)$$

Hence from (1.31)

$$u' = -\frac{\lambda_1}{\eta}f'(\eta)\bar{x} - \frac{\lambda_2}{\eta}\phi'(\eta)\bar{x} + F(\eta) \quad \ldots \quad (2.2)$$

where $f(\eta)$, $\phi(\eta)$ and $F(\eta)$ are unknown functions to be determined.
Now substituting the values of v' and u' from relations (2.1) and (2.2) in eqns. (1.28) and (1.29), we get

$$\begin{aligned}\frac{\partial p'}{\partial \bar{x}} = {} & F'' + \frac{1}{\eta}F' - \frac{1}{\eta}[\lambda_1(F'f - f'F) + \lambda_2(F'\phi - F\phi')] \\ & - \frac{\bar{x}}{\eta^3}[\lambda_1(\eta^2 f''' - \eta f'' + f') + \lambda_2(\eta^2\phi''' - \eta\phi'' + \phi') \\ & + \lambda_1^2(-\eta f f'' + f f' + \eta f'^2) + \lambda_2^2(-\eta\phi\phi'' + \phi\phi' + \eta\phi'^2) \\ & + \lambda_1\lambda_2(-\eta f\phi'' + f\phi' - \eta\phi f'' + \phi f' + 2\eta f'\phi')] \quad \ldots \quad (2.3)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial p'}{\partial \eta} = {} & \frac{2w_0 w'}{\eta} + \frac{w'^2}{\eta} + \frac{1}{\eta^3}[\lambda_1(\eta^2 f'' - \eta f') + \lambda_2(\eta^2\phi'' - \eta\phi') \\ & - \lambda_1^2(\eta f f' - f^2) - \lambda_2^2(\eta\phi\phi' - \phi^2) + \lambda_1\lambda_2(\eta f\phi' - 2f\phi + \eta\phi f')] \quad \ldots \quad (2.4)\end{aligned}$$

with the boundary conditions

$$\left.\begin{aligned}\eta = \sigma: f(\eta) = 1,\ \phi(\eta) = 0,\ f'(\eta) = 0,\ \phi'(\eta) = 0,\ F(\eta) = 0 \\ \eta = 1: f(\eta) = 0,\ \phi(\eta) = 1,\ f'(\eta) = 0,\ \phi'(\eta) = 0,\ F(\eta) = 0\end{aligned}\right\}. \quad \ldots \quad (2.5)$$

Now at $\bar{x} = 0$, if we maintain the pressure gradient equal to the pressure gradient without porosity, then

$$\text{at } \bar{x} = 0,\ \frac{\partial p'}{\partial \bar{x}} = 0.$$

Hence from (2.3), we have

$$F'' + \frac{1}{\eta}F' - \frac{1}{\eta}[\lambda_1(F'f - Ff') + \lambda_2(F'\phi - F\phi')] = 0. \quad \ldots \quad (2.6)$$

Now since the right-hand side of eqn. (2.4) is purely a function of η, we have

$$\frac{\partial^2 p'}{\partial \bar{x}\partial \eta} = 0. \quad \ldots \quad (2.7)$$

Now using (2.6) and (2.7) in relation (2.3), we get

$$\begin{aligned}& \lambda_1(\eta^3 f''' - \eta^2 f'' + \eta f') + \lambda_2(\eta^3\phi''' - \eta^2\phi'' + \eta\phi') \\ & + \lambda_1^2(\eta^2 f'^2 - \eta^2 f f'' + \eta f f') + \lambda_2^2(\eta^2\phi'^2 - \eta^2\phi\phi'' + \eta\phi\phi') \\ & + \lambda_1\lambda_2(2\eta^2 f'\phi' - \eta^2 f\phi'' + \eta f\phi' - \eta^2\phi f'' + \eta\phi f') = K\eta^4, \quad \ldots \quad (2.8)\end{aligned}$$

where K is the constant of integration to be determined.

PERTURBATION SOLUTION FOR SMALL λ_1 AND λ_2 $(0 \leqslant \lambda_1 < 1;\ 0 \leqslant \lambda_2 < 1)$

For the solution of eqn. (2.8) we express a power series for small values of λ_1 and λ_2 near $\lambda_1 = 0 = \lambda_2$ as

$$f(\eta) = f_0(\eta) + \lambda_1 f_1(\eta) + \lambda_1^2 f_2(\eta) + \ldots + \lambda_1^n f_n(\eta) + \ldots \quad (3.1)$$

$$\phi(\eta) = \phi_0(\eta) + \lambda_2 \phi_1(\eta) + \lambda_2^2 \phi_2(\eta) + \ldots + \lambda_2^n \phi_n(\eta) + \ldots \quad (3.2)$$

and

$$K = K_0 + \lambda_1 K_1 + \lambda_2 K'_1 + \lambda_1^2 K_2 + \lambda_2^2 K'_2 + \ldots + \lambda_1^n K_n + \lambda_2^n K'_n + \ldots \quad (3.3)$$

In relations (3.1) and (3.2) all the f_n's and ϕ_n's are taken to be independent of λ_1 and λ_2.

Now substituting (3.1), (3.2) and (3.3) in relation (2.8) and equating like powers of λ_1 and λ_2, we get

$$K_0 = 0 \quad (3.4)$$

$$\eta^3 f_0''' - \eta^2 f_0'' + \eta f_0' = K_1 \eta^4 \quad (3.5)$$

$$\eta^3 f_1''' - \eta^2 f_1'' + \eta f_1' - \eta^2 f_0 f_0'' + \eta f_0 f_0' + \eta^2 f_0'^2 = K_2 \eta^4 \quad (3.6)$$

$$\eta^3 f_2''' - \eta^2 f_2'' + \eta f_2' - \eta^2 f_1 f_0'' - \eta^2 f_0 f_1'' + \eta f_0 f_1' + \eta f_1 f_0' + 2\eta^2 f_0' f_1' = K_3 \eta^4 \quad (3.7)$$

with the boundary conditions

$$\left.\begin{array}{l} f_0(\sigma) = 1,\ f_n(\sigma) = 0 \text{ for } n \geqslant 1,\ f_n'(\sigma) = 0 \text{ for } n \geqslant 0 \\ f_n(1) = 0 \text{ and } f_n'(1) = 0 \text{ for } n \geqslant 0 \end{array}\right\} \quad (3.8)$$

and

$$\eta^3 \phi_0''' - \eta^2 \phi_0'' + \eta \phi_0' = K_1' \eta^4 \quad (3.9)$$

$$\eta^3 \phi_1''' - \eta^2 \phi_1'' + \eta \phi_1' - \eta^2 \phi_0 \phi_0'' + \eta \phi_0 \phi_0' + \eta^2 \phi_0'^2 = K_2' \eta^4 \quad (3.10)$$

$$\eta^3 \phi_2''' - \eta^2 \phi_2'' + \eta \phi_2' - \eta^2 \phi_1 \phi_0'' - \eta^2 \phi_0 \phi_1'' + \eta \phi_0 \phi_1' + \eta \phi_1 \phi_0' + 2\eta^2 \phi_0' \phi_1' = K_3' \eta^4 \quad (3.11)$$

with boundary conditions

$$\left.\begin{array}{l} \phi_n(\sigma) = 0 \text{ and } \phi_n'(\sigma) = 0 \text{ for } n \geqslant 0 \\ \phi_0(1) = 1,\ \phi_n(1) = 0 \text{ for } n \geqslant 1,\ \phi_n'(1) = 0 \text{ for } n \geqslant 0 \end{array}\right\}. \quad (3.12)$$

Equations (3.5) and (3.8) give

$$f_0(\eta) = A_1 + (A_2 + A_3 \log \eta)\eta^2 + \frac{1}{16} K_1 \eta^4 \quad (3.13)$$

where

$$A_1 = \frac{K_1(1-\sigma^2)}{16 \log \sigma} + \frac{K_1}{16} \quad (3.14)$$

$$A_2 = \frac{K_1(\sigma^2-1)}{16 \log \sigma} - \frac{K_1}{8} \quad (3.15)$$

$$A_3 = \frac{K_1(1-\sigma^2)}{8 \log \sigma} \quad (3.16)$$

and

$$K_1 = \frac{16 \log \sigma}{(1-\sigma^2)[(1-\sigma^2)+(1+\sigma^2)\log \sigma]}. \qquad \ldots \quad \ldots \quad (3.17)$$

The equation for $f_1(\eta)$ is

$$\eta^3 f_1''' - \eta^2 f_1'' + \eta f_1' = K_2\eta^4 + \eta^2 f_0 f_0'' - \eta f_0 f_0' - \eta^2 f_0'^2. \qquad \ldots \quad (3.18)$$

The solution of (3.18) using (3.13) is

$$\begin{aligned} f_1(\eta) = {} & B_1 + (B_2 \log \eta + B_3)\eta^2 + K_2\frac{\eta^4}{16} - \frac{K_1^2}{256\times 36}\eta^8 \\ & - \frac{K_1}{96\times 24}(12A_2 + A_3 + 12A_3 \log \eta)\eta^6 + \frac{1}{32}[K_1A_1 + 16A_2A_3 - 8A_2^2 - 14A_3^2 \\ & - 16A_2A_3 \log \eta + 16A_3^2 \log \eta - 8A_3^2(\log \eta)^2]\eta^4 + \tfrac{1}{4}A_1A_3[2(\log \eta)^2 \\ & - 2 \log \eta + 1]\eta^2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.19) \end{aligned}$$

and

$$\begin{aligned} f'_1(\eta) = {} & (2B_2 \log \eta + 2B_3 + B_2)\eta + \frac{K_2}{4}\eta^3 - \frac{K_1^2}{32\times 36}\eta^7 - \frac{K_1}{128}[4A_2 + A_3 + 4A_3 \log \eta]\eta^5 \\ & + \tfrac{1}{8}[K_1A_1 + 12A_2A_3 - 8A_2^2 - 10A_3^2 - 16A_2A_3 \log \eta + 12A_3^2 \log \eta \\ & - 8A_3^2(\log \eta)^2]\eta^3 + A_1A_3(\log \eta)^2\eta \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.20) \end{aligned}$$

where B_1, B_2, B_3 and K_2 are constants of integration to be determined by the following four boundary conditions:

$$\left. \begin{aligned} f_1(1) = 0, \quad f'_1(1) = 0 \\ f_1(\sigma) = 0, \quad f'_1(\sigma) = 0 \end{aligned} \right\}. \qquad \ldots \quad \ldots \quad \ldots \quad (3.21)$$

The boundary condition $f_1(1) = 0$ gives

$$\begin{aligned} B_1 + B_3 + \frac{K_2}{16} & = \frac{K_1^2}{256\times 36} + \frac{K_1}{96\times 24}(12A_2 + A_3) - \frac{1}{32}(K_1A_1 + 16A_2A_3 - 8A_2^2 - 14A_3^2) \\ & \quad - \tfrac{1}{4}A_1A_3 \\ & = T \text{ (say)}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.22) \end{aligned}$$

The boundary condition $f'_1(1) = 0$ gives

$$\begin{aligned} 2B_3 + B_2 + \frac{K_2}{4} & = \frac{K_1^2}{32\times 36} + \frac{K_1}{128}(4A_2 + A_3) - \tfrac{1}{8}[K_1A_1 + 12A_2A_3 - 8A_2^2 - 10A_3^2] \\ & = U \text{ (say)}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.23) \end{aligned}$$

The boundary condition $f_1(\sigma) = 0$ gives

$$\begin{aligned} B_1 + (B_2 \log \sigma + B_3)\sigma^2 + \frac{K_2}{16}\sigma^4 & = \frac{K_1^2}{256\times 36}\sigma^8 + \frac{K_1}{96\times 24}(12A_2 + A_3 + 12A_3 \log \sigma)\sigma^6 \\ & \quad - \frac{1}{32}[K_1A_1 + 16A_2A_3 - 8A_2^2 - 14A_3^2 - 16A_2A_3 \log \sigma \\ & \quad + 16A_3^2 \log \sigma - 8A_3^2 (\log \sigma)^2]\sigma^4 - \tfrac{1}{4}A_1A_3[2(\log \sigma)^2 \\ & \quad - 2(\log \sigma) + 1]\sigma^2 \\ & = V \text{ (say)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.24) \end{aligned}$$

and the boundary condition $f'_1(\sigma) = 0$ gives

$$(2B_2 \log \sigma + 2B_3 + B_2)\sigma + \frac{K_2}{4}\sigma^3 = \frac{K_1^2}{32\times 36}\sigma^7 + \frac{K_1}{128}[4A_2 + A_3 + 4A_3 \log \sigma]\sigma^5$$

$$-\tfrac{1}{8}[K_1A_1 + 12A_2A_3 - 8A_2^2 - 10A_3^2 - 16A_2A_3 \log \sigma$$

$$+12A_3^2 \log \sigma - 8A_3^2 (\log \sigma)^2]\sigma^3 - A_1A_3 (\log \sigma)^2\sigma$$

$$= W \text{ (say)}. \qquad (3.25)$$

Solving (3.22) to (3.25), we get

$$B_1 = \frac{2\sigma \log \sigma(2T-U) + (W-U\sigma)}{4\sigma \log \sigma} + \frac{\log \sigma + 1 - \sigma^2}{16 \log \sigma} K_2 \qquad (3.26)$$

$$B_2 = \frac{W-U\sigma}{2\sigma \log \sigma} + \frac{K_2(1-\sigma^2)}{8 \log \sigma} \qquad (3.27)$$

$$B_3 = \frac{U\sigma(1+2 \log \sigma) - W}{4\sigma \log \sigma} - \frac{(2 \log \sigma + 1 - \sigma^2)K_2}{16 \log \sigma} \qquad (3.28)$$

and

$$K_2 = \frac{4[4\sigma \log \sigma(V-T) + (1-\sigma^2)(\sigma U - W) + 2\sigma \log \sigma(U - \sigma W)]}{\sigma(1-\sigma^2)[(1-\sigma^2) + (1+\sigma^2) \log \sigma]}. \qquad (3.29)$$

The equation for $f_2(\eta)$ is

$$\eta^3 f_2''' - \eta^2 f_2'' + \eta f_2' = K_3\eta^4 + \eta^2 f_1 f_0'' + \eta^2 f_0 f_1'' - \eta f_0 f_1' - \eta f_1 f_0' - 2\eta^2 f_0' f_1'. \qquad (3.30)$$

The solution of (3.30) using (3.13) and (3.19) is

$$f_2(\eta) = D_1 + (D_2 \log \eta + D_3)\eta^2 + \frac{K_3}{16}\eta^4 + \frac{K_1^3}{256\times 86400}\eta^{12} + \frac{K_1^2}{256\times 72\times 640}$$

$$\times(\psi_1 + \chi_2 \log \eta)\eta^{10} + \frac{K_1}{96\times 72\times 8\times 36}[\psi_2 + \psi_3 \log \eta + \chi_5 (\log \eta)^2]\eta^8$$

$$+\frac{1}{16\times 24\times 96}[\psi_4 + \psi_5 \log \eta + \psi_6 (\log \eta)^2 + \chi_9 (\log \eta)^3]\eta^6$$

$$-\frac{1}{128}[\psi_7 + \psi_8 \log \eta + \psi_9 (\log \eta)^2 + \chi_{13} (\log \eta)^3]\eta^4 + \Big[\{4 - 2 \log \eta$$

$$+2 (\log \eta)^2\}\psi_{10} + \frac{1}{12}\chi_{15} (\log \eta)^3\Big]\eta^2 \qquad (3.31)$$

and

$$f'_2(\eta) = (2D_2 \log \eta + 2D_3 + D_2)\eta + \frac{K_3}{4}\eta^3 + \frac{K_1^3}{256\times 7200}\eta^{11}$$

$$+\frac{K_1^2}{256\times 72\times 640}[10\psi_1 + \chi_2 + 10\chi_2 \log \eta]\eta^9 + \frac{K_1}{96\times 72\times 288}[8\psi_2 + \psi_3$$

$$+(8\psi_3 + 2\chi_5) \log \eta + 8\chi_5 (\log \eta)^2]\eta^7 + \frac{1}{384\times 96}[6\psi_4 + \psi_5 + (6\psi_5 + 2\psi_6) \log \eta$$

$$+(6\psi_6 + 3\chi_9)(\log \eta)^2 + 6\chi_9 (\log \eta)^3]\eta^5 - \frac{1}{128}[4\psi_7 + \psi_8$$

$$+(4\psi_8 + 2\psi_9) \log \eta + (4\psi_9 + 3\chi_{13})(\log \eta)^2 + 4\chi_{13} (\log \eta)^3]\eta^3$$

$$+\Big[\{6 + 4 (\log \eta)^2\}\psi_{10} + \frac{1}{12}\chi_{15}\{2 (\log \eta)^3 + 3 (\log \eta)^2\}\Big]\eta \qquad (3.32)$$

where D_1, D_2, D_3, and K_3 are constants of integration and all the χ's and ψ's are given by

$$\chi_1 = 64A_2 - 6A_3 \quad (3.33)$$

$$\chi_2 = 64A_3 \quad (3.34)$$

$$\chi_3 = 1728A_2^2 - 2880A_2A_3 + 2478A_3^2 - 432K_2 - 252K_1A_1 \quad (3.35)$$

$$\chi_4 = 3456A_2A_3 - 2880A_3^2 \quad (3.36)$$

$$\chi_5 = 1728A_3^2 \quad (3.37)$$

$$\chi_6 = -192A_2K_2 - 144A_2K_1A_1 - 192A_2^2A_3 + 768A_2^3 + 768A_2A_3^2 - 144A_3K_2 - 48A_3K_1A_1 + 624A_3^3 - 192K_1B_1 - 144K_1B_2 \quad (3.38)$$

$$\chi_7 = 2304A_2^2A_3 - 2688A_3^2A_2 - 192A_3K_2 - 192A_3K_1A_1 + 768A_3^3 - 192K_1B_2 \quad (3.39)$$

$$\chi_8 = 2304A_2A_3^2 - 1344A_3^3 - 96K_1A_1A_3 \quad (3.40)$$

$$\chi_9 = 768A_3^3 \quad (3.41)$$

$$\chi_{10} = 64A_2B_3 + 16A_2B_2 + 16A_3B_3 + 16A_3B_2 - 4B_1K_1 - 8A_1A_2A_3 + 4A_1A_3^2 - 4A_1K_1 - 2A_1^2K_1 + 16A_1A_2^2 \quad (3.42)$$

$$\chi_{11} = 32A_1A_2A_3 + 64A_3B_2 + 8A_3^2A_1 \quad (3.43)$$

$$\chi_{12} = 64A_2B_2 + 64A_3B_3 + 32A_3B_2 + 16A_1A_2A_3 \quad (3.44)$$

$$\chi_{13} = 32A_1A_3^2 \quad (3.45)$$

$$\chi_{14} = 2(B_1A_3 + A_1B_2) \quad (3.46)$$

$$\chi_{15} = 2A_1^2A_3 \quad (3.47)$$

and

$$\psi_1 = \chi_1 - \frac{7}{20}\chi_2 \quad (3.48)$$

$$\psi_2 = \chi_3 - \frac{11}{20}\chi_4 + \frac{81}{288}\chi_5 \quad (3.49)$$

$$\psi_3 = \chi_4 - \frac{11}{12}\chi_5 \quad (3.50)$$

$$\psi_4 = \chi_6 - \frac{2}{3}\chi_7 + \frac{43}{72}\chi_8 - \frac{97}{144}\chi_9 \quad (3.51)$$

$$\psi_5 = \chi_7 - \frac{4}{3}\chi_8 + \frac{43}{24}\chi_9 \quad (3.52)$$

$$\psi_6 = \chi_8 - 2\chi_9 \quad (3.53)$$

$$\psi_7 = \chi_{10} - \frac{5}{4}\chi_{12} + \frac{17}{8}\chi_{11} - \frac{147}{32}\chi_{13} \quad (3.54)$$

$$\psi_8 = \chi_{12} - \frac{5}{2}\chi_{11} + \frac{51}{8}\chi_{13} \quad (3.55)$$

$$\psi_9 = \chi_{11} - \frac{15}{4}\chi_{13} \quad (3.56)$$

$$\psi_{10} = 2\chi_{14} - \chi_{15}. \quad (3.57)$$

The boundary conditions for the constants D_1, D_2, D_3 and K_3 are:

$$\left.\begin{array}{l} f_2(1) = 0, \quad f_2'(1) = 0 \\ f_2(\sigma) = 0, \quad f_2'(\sigma) = 0 \end{array}\right\}. \qquad \ldots \quad \ldots \quad \ldots (3.58)$$

The boundary condition $f_2(1) = 0$ gives

$$D_1+D_3+\frac{K_3}{16}+\frac{K_1^3}{256\times 86400}+\frac{K_1^2}{256\times 72\times 640}\psi_1+\frac{K_1}{96\times 72\times 288}\psi_2$$

$$+\frac{1}{384\times 96}\psi_4-\frac{1}{128}\psi_7+4\psi_{10} = 0$$

or

$$D_1+D_3+\frac{K_3}{16} = P \text{ (say)}. \qquad \ldots \quad \ldots \quad \ldots (3.59)$$

The boundary condition $f_2(\sigma) = 0$ gives

$$D_1+(D_2 \log \sigma+D_3)\sigma^2+\frac{K_3}{16}\sigma^4+\frac{K_1^3}{256\times 86400}\sigma^{12}+\frac{K_1^2}{256\times 72\times 640}$$

$$\times(\psi_1+\chi_2 \log \sigma)\sigma^{10}+\frac{K_1}{96\times 72\times 288}\{\psi_2+\psi_3 \log \sigma+\chi_5(\log \sigma)^2\}\sigma^8+\frac{1}{384\times 96}$$

$$\times\{\psi_4+\psi_5 \log \sigma+\psi_6 (\log \sigma)^2+\chi_9 (\log \sigma)^3\}\sigma^6-\frac{1}{128}\{\psi_7+\psi_8 \log \sigma$$

$$+\psi_9 (\log \sigma)^2+\chi_{13} (\log \sigma)^3\}\sigma^4+\left[\{4-2 \log \sigma+2 (\log \sigma)^2\}\psi_{10}+\frac{1}{12}\chi_{15} (\log \sigma)^3\right]\sigma^2 = 0$$

or

$$D_1+(D_2 \log \sigma+D_3)\sigma^2+\frac{1}{16}K_3\sigma^4 = Q \text{ (say)}. \qquad \ldots \quad \ldots (3.60)$$

The boundary condition $f'_2(1) = 0$ gives

$$2D_3+D_2+\frac{K_3}{4}+\frac{K_1^3}{256\times 7200}+\frac{K_1^2}{256\times 72\times 640}[10\psi_1+\chi_2]$$

$$+\frac{K_1}{96\times 72\times 288}[8\psi_2+\psi_3]+\frac{1}{384\times 96}[6\psi_4+\psi_5]-\frac{1}{128}[4\psi_7+\psi_8]+6\psi_{10} = 0$$

or

$$2D_3+D_2+\tfrac{1}{4}K_3 = R \text{ (say)} \qquad \ldots \quad \ldots \quad \ldots (3.61)$$

and the boundary condition $f'_2(\sigma) = 0$ gives

$$(2D_2 \log \sigma+2D_3+D_2)\sigma+\frac{K_3}{4}\sigma^3+\frac{K_1^3}{256\times 7200}\sigma^{11}+\frac{K_1^2}{256\times 72\times 640}[10\psi_1+\chi_2$$

$$+10\chi_2 \log \sigma]\sigma^9+\frac{K_1}{96\times 72\times 288}[8\psi_2+\psi_3+(8\psi_3+2\chi_5) \log \sigma+8\chi_5 (\log \sigma)^2]\sigma^7$$

$$+\frac{1}{384\times 96}[6\psi_4+\psi_5+(6\psi_5+2\psi_6) \log \sigma+(6\psi_6+3\chi_9) (\log \sigma)^2+6\chi_9 (\log \sigma)^3]\sigma^5$$

$$-\frac{1}{128}[4\psi_7+\psi_8+(4\psi_8+2\psi_9) \log \sigma+(4\psi_9+3\chi_{13}) (\log \sigma)^2+4\chi_{13} (\log \sigma)^3]\sigma^3$$

$$+[\{6+4 (\log \sigma)^2\}\psi_{10}+\frac{1}{12}\chi_{15}\{2 (\log \sigma)^3+3 (\log \sigma)^2\}]\sigma = 0$$

or

$$2(D_2 \log \sigma+2D_3+D_2)\sigma+\frac{K_3}{4}\sigma^3 = S \text{ (say)}. \quad \ldots \quad \ldots \quad (3.62)$$

Solving (3.59) to (3.62), we get

$$D_1 = \frac{2\sigma \log \sigma(2P-R)+(S-R\sigma)}{4\sigma \log \sigma}+\frac{\log \sigma+1-\sigma^2}{16 \log \sigma}K_3 \quad \ldots \quad (3.63)$$

$$D_2 = \frac{S-R\sigma}{2\sigma \log \sigma}+\frac{K_3(1-\sigma^2)}{8 \log \sigma} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.64)$$

$$D_3 = \frac{R\sigma(1+2 \log \sigma)-S}{4\sigma \log \sigma}-\frac{(2 \log \sigma+1-\sigma^2)}{16 \log \sigma}K_3 \quad \ldots \quad \ldots \quad (3.65)$$

and

$$K_3 = \frac{4[4\sigma \log \sigma(Q-P)+(1-\sigma^2)(\sigma R-S)+2\sigma \log \sigma(R-\sigma S)]}{\sigma(1-\sigma^2)[(1-\sigma^2)+(1+\sigma^2) \log \sigma]}. \quad \ldots \quad (3.66)$$

Hence the second order perturbation solution for $f(\eta)$ satisfying the boundary conditions (3.8) is

$$f(\eta) = f_0(\eta)+\lambda_1 f_1(\eta)+\lambda_1^2 f_2(\eta) \quad \ldots \quad \ldots \quad \ldots \quad (3.67)$$

where $f_0(\eta)$, $f_1(\eta)$ and $f_2(\eta)$ are given by (3.13), (3.19) and (3.31) respectively.

Equations (3.9) and (3.12) give

$$\phi_0(\eta) = A'_1+(A'_2+A'_3 \log \eta)\eta^2+\frac{K'_1}{16}\eta^4 \quad \ldots \quad \ldots \quad (3.68)$$

where

$$A'_1 = 1+\frac{K'_1(1-\sigma^2)}{16 \log \sigma}+\frac{K'_1}{16} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.69)$$

$$A'_2 = \frac{K'_1(\sigma^2-1)}{16 \log \sigma}-\frac{K'_1}{8} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.70)$$

$$A'_3 = \frac{K'_1(1-\sigma^2)}{8 \log \sigma} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.71)$$

and

$$K'_1 = \frac{-16 \log \sigma}{(1-\sigma^2)[(1-\sigma^2)+(1+\sigma^2) \log \sigma]}. \quad \ldots \quad \ldots \quad (3.72)$$

The equation for (η) is

$$\eta^3\phi_1'''-\eta^2\phi_1''+\eta\phi_1' = K_2'\eta^4+\eta^2\phi_0\phi_0''-\eta\phi_0\phi_0'-\eta^2\phi_0'^2. \quad \ldots \quad \ldots \quad (3.73)$$

The solution of (3.73) using (3.68) is

$$\phi_1(\eta) = B_1'+(B_2' \log \eta+B_3')\eta^2+\frac{K_2'}{16}\eta^4-\frac{K_1'^2}{256\times 36}\eta^8-\frac{K_1'}{96\times 24}$$
$$\times(12A_2'+A_3'+12A_3' \log \eta)\eta^6+\frac{1}{32}[K_1'A_1'+16A_2'A_3'-8A_2'^2-14A_3'^2-16A_2'A_3'$$
$$\times \log \eta+16A_3'^2 \log \eta-8A_3'^2 (\log \eta)^2]\eta^4+\tfrac{1}{4}A_1'A_3'[2 (\log \eta)^2-2 \log \eta+1]\eta^2$$
$$\ldots \quad (3.74)$$

and

$$\phi_1'(\eta) = \left(2B_2' \log \eta + 2B_3' + B_2'\right)\eta + \frac{K_2'}{4}\eta^3 - \frac{K_1'^2}{32\times 36}\eta^7 - \frac{K_1'}{128}\left[4A_2' + A_3' + 4A_3' \log \eta\right]\eta^5 + \tfrac{1}{8}\left[K_1'A_1' + 12A_2'A_3' - 8A_2'^2 - 10A_3'^2 - 16A_2'A_3' \log \eta + 12A_3'^2 \log \eta - 8A_3'^2 (\log \eta)^2\right]\eta^3 + A_1'A_3' (\log \eta)^2\eta \quad \ldots \quad (3.75)$$

where B'_1, B'_2, B'_3 and K'_2 are constants of integration to be determined by the following four boundary conditions:

$$\left.\begin{array}{l} \phi_1(1) = 0, \quad \phi_1'(1) = 0 \\ \phi_1(\sigma) = 0, \quad \phi_1'(\sigma) = 0 \end{array}\right\}. \quad \ldots \quad (3.76)$$

The boundary condition $\phi_1(1) = 0$ gives

$$B_1' + B_3' + \frac{K_2'}{16} = \frac{K_1'^2}{256\times 36} + \frac{K_1'}{96\times 24}(12A_2' + A_3') - \frac{1}{32}\left(K_1'A_1' + 16A_2'A_3' - 8A_2'^2 - 14A_3'^2\right) - \tfrac{1}{4}A_1'A_3'$$
$$= T'(\text{say}). \quad \ldots \quad (3.77)$$

The boundary condition $\phi_1(1) = 0$ gives

$$2B'_4 + B'_4 + \frac{K_2'}{4} = \frac{K_1'^2}{32\times 36} + \frac{K_1'}{128}(+A'_2 + A'_3) - \tfrac{1}{8}\left(K_1'A_1' + 12A_2'A_3' - 8A_2'^2 - 10A_3'^2\right)$$
$$= U'(\text{say}). \quad \ldots \quad (3.78)$$

The boundary condition $\phi_1(\sigma) = 0$ gives

$$B_1' + \left(B_2' \log \sigma + B_3'\right)\sigma^2 + \frac{K_2'}{16}\sigma^4 = \frac{K_1'^2}{256\times 36}\sigma^8 + \frac{K_1'}{96\times 24}\left(12A_2' + A_3' + 12A_3' \log \sigma\right)\sigma^6 - \frac{1}{32}\left[K_1'A_1' + 16A_2'A_3' - 8A_2'^2 - 14A_3'^2 - 16A_2'A_3' \log \sigma + 16A_3'^2 \log \sigma - 8A_3'^2 (\log \sigma)^2\right]\sigma^4 - \frac{1}{4}A'_1A'_3[2(\log \sigma)^2 - 2\log \sigma + 1]\sigma^2$$
$$= V' \text{ (say)} \quad \ldots \quad (3.79)$$

and the boundary condition $\phi'_1(\sigma) = 0$ gives

$$\left(2B_2' \log \sigma + 2B_3' + B_2'\right)\sigma + \frac{K_2'}{4}\sigma^3 = \frac{K_1'^2}{32\times 36}\sigma^7 + \frac{K_1'}{128}[4A'_2 + A'_3 + 4A'_3 \log \sigma]\sigma^5 - \tfrac{1}{8}\left[K_1'A_1' + 12A_2'A_3' - 8A_2'^2 - 10A_3'^2 - 16A_2'A_3' \log \sigma \; 12A_3'^2 \log \sigma - 8A_3'^2 (\log \sigma)^2\right]\sigma^3 - A_1'A_3' (\log \sigma)^2\sigma$$
$$= W' \text{ (say)}. \quad \ldots \quad (3.80)$$

Solving (3.77) to (3.80), we get

$$B'_1 = \frac{2\sigma \log \sigma(2T'-U')+(W'-U'\sigma)}{4\sigma \log \sigma} + \frac{\log \sigma + 1 - \sigma^2}{16 \log \sigma} K'_2 \quad \ldots \quad \ldots \quad (3.81)$$

$$B'_2 = \frac{W'-U'\sigma}{2\sigma \log \sigma} + \frac{K'_2(1-\sigma^2)}{8 \log \sigma} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.82)$$

$$B'_3 = \frac{U'\sigma(1+2\log \sigma)-W'}{4\sigma \log \sigma} - \frac{(2\log \sigma + 1 - \sigma^2)}{16 \log \sigma} K'_2 \quad \ldots \quad \ldots \quad \ldots \quad (3.83)$$

$$K'_2 = \frac{4[4\sigma \log \sigma(V'-T')+(1-\sigma^2)(\sigma U'-W')+2\sigma \log \sigma(U'-\sigma W')]}{\sigma(1-\sigma^2)[(1-\sigma^2)+(1+\sigma^2)\log \sigma]}. \quad (3.84)$$

The equation for $\phi_2(\eta)$ is

$$\eta^3\phi_2''' - \eta^2\phi_2'' + \eta\phi_2' = K_3'\eta^4 + \eta^2\phi_1\phi_0'' + \eta^2\phi_0\phi_1'' - \eta\phi_0\phi_1' - \eta\phi_1\phi_0' - 2\eta^2\phi_0'\phi_1'. \quad (3.85)$$

The solution of (3.85) using (3.68) and (3.74) is

$$\phi_2(\eta) = D_1' + (D_2' \log \eta + D_3')\eta^2 + \frac{K_3'}{16}\eta^4 + \frac{K_1'^3}{256\times 86400}\eta^{12} + \frac{K_1'^2}{256\times 72\times 640}$$
$$\times(\psi_1' + \chi_2' \log \eta)\eta^{10} + \frac{K_1'}{96\times 72\times 288}[\psi_2' + \psi_3' \log \eta + \chi_5' (\log \eta)^2]\eta^8$$
$$+ \frac{1}{384\times 96}[\psi_4' + \psi_5' \log \eta + \psi_6' (\log \eta)^2 + \chi_9' (\log \eta)^3]\eta^6 - \frac{1}{128}$$
$$\times[\psi_7' + \psi_8' \log \eta + \psi_9' (\log \eta)^2 + \chi_{13}' (\log \eta)^3]\eta^4 + \Big[\{4 - 2\log \eta$$
$$+ 2(\log \eta)^2\}\psi_{10}' + \frac{1}{12}\chi_{15}' (\log \eta)^3\Big]\eta^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.86)$$

and

$$\phi_2'(\eta) = (2D_2' \log \eta + 2D_3' + D_2')\eta + \frac{K_3'}{4}\eta^3 + \frac{K_1'^3}{256\times 7200}\eta^{11} + \frac{K_1'^2}{256\times 72\times 640}$$
$$\times[10\psi_1' + \chi_2' + 10\chi_2' \log \eta]\eta^9 + \frac{K_1'}{96\times 72\times 288}[8\psi_2' + \psi_3' + (8\psi_3' + 2\chi_5') \log \eta$$
$$+ 8\chi_5' (\log \eta)^2]\eta^7 + \frac{1}{384\times 96}[6\psi_4' + \psi\psi_5' + (6\psi_5' + 2\chi_6') \log \eta$$
$$+ (6\psi_6' + 3\chi_9') (\log \eta)^2 + 6\chi_9' (\log \eta)^3]\eta^5 - \frac{1}{128}[4\psi_7' + \psi_8'$$
$$+ (4\psi_8' + 2\psi_9') \log \eta + (4\psi_9' + 3\chi_{13}')(\log \eta)^2 + 4\chi_{13}'(\log \eta)^3]\eta^3$$
$$+ \Big[\{6 + 4(\log \eta)^2\}\psi_{10}' + \frac{1}{12}\chi_{15}'\{2(\log \eta)^3 + 3(\log \eta)^2\}\Big]\eta \quad \ldots \quad \ldots \quad (3.87)$$

where D'_1, D'_2, D'_3 and K'_3 are constants of integration.

The (χ')'s and (ψ')'s are as below:

$$\chi'_1 = 64A'_2 - 6A'_3 \quad (3.88)$$

$$\chi'_2 = 64A'_3 \quad (3.89)$$

$$\chi'_3 = 1728A_2'^2 - 2880A'_2A'_3 + 2478A_3'^2 - 432K'_2 - 252K'_1A'_1 \quad (3.90)$$

$$\chi'_4 = 3456A'_2A'_3 - 2880A_3'^2 \quad (3.91)$$

$$\chi'_5 = 1728A_3'^2 \quad (3.92)$$

$$\chi'_6 = -192A'_2K'_2 - 144A'_2K'_1A'_1 - 192A'_2A'_3 + 768A_2'^3 + 768A'_2A_3'^2$$
$$-144A'_3K'_2 - 48A'_3K'_1A'_1 + 624A_3'^2 - 192K'_1B'_3 - 144K'_1B'_2 \quad (3.93)$$

$$\chi'_7 = 2304A_2'^2A'_3 - 2688A_3'^2A'_2 - 192A'_3K'_2 - 192A'_3K'_1A'_1$$
$$+768A_3'^3 - 192K'_1B'_2 \quad (3.94)$$

$$\chi'_8 = 2304A'_2A_3'^2 - 1344A_3'^3 - 96K'_1A'_1A'_3 \quad (3.95)$$

$$\chi'_9 = 768A_3'^3 \quad (3.96)$$

$$\chi'_{10} = 64A'_2B'_3 + 16A'_2B'_2 + 16A'_3B'_3 + 16A'_3B'_2 - 4B'_1K'_1 - 8A'_1A'_2A'_3$$
$$+4A'_1A_3'^2 - 4A'_1K'_2 - 2A_1'^2K'_1 + 16A'_1A_2'^2 \quad (3.97)$$

$$\chi'_{11} = 32A'_1A'_2A'_3 + 64A'_3B'_2 + 8A_3'^2A'_1 \quad (3.98)$$

$$\chi'_{12} = 64A'_2B'_2 + 64A'_3B'_3 + 32A'_3B'_2 + 16A'_1A'_2A'_3 \quad (3.99)$$

$$\chi'_{13} = 32A'_1A_3'^2 \quad (3.100)$$

$$\chi'_{14} = 2(B'_1A'_3 + A'_1B'_2) \quad (3.101)$$

$$\chi'_{15} = 2A_1'^2A'_3 \quad (3.102)$$

and

$$\psi'_1 = \chi'_1 - \frac{7}{20}\chi'_2 \quad (3.103)$$

$$\psi'_2 = \chi'_3 - \frac{11}{20}\chi'_4 + \frac{81}{288}\chi'_5 \quad (3.104)$$

$$\psi'_3 = \chi'_4 - \frac{11}{12}\chi'_5 \quad (3.105)$$

$$\psi'_4 = \chi'_6 - \frac{2}{3}\chi'_7 + \frac{43}{72}\chi'_8 - \frac{97}{144}\chi'_9 \quad (3.106)$$

$$\psi'_5 = \chi'_7 - \frac{4}{3}\chi'_8 + \frac{43}{28}\chi'_9 \quad (3.107)$$

$$\psi'_6 = \chi'_8 - 2\chi'_9 \quad (3.108)$$

$$\psi_7' = \chi_{10}' - \frac{5}{4}\chi_{12}' + \frac{17}{8}\chi_{11}' - \frac{147}{32}\chi_{13}' \qquad \ldots \quad \ldots (3.109)$$

$$\psi_8' = \chi_{12}' - \frac{5}{2}\chi_{11}' + \frac{51}{8}\chi_{13}' \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (3.110)$$

$$\psi_9' = \chi_{11}' - \frac{15}{4}\chi_{13}' \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.111)$$

$$\psi_{10}' = 2\chi_{14}' - \chi_{15}'. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.112)$$

The boundary conditions for the constants D'_1, D'_3, D'_3 and K'_3 are

$$\left.\begin{array}{ll} \phi_2(1) = 0, & \phi'_2(1) = 0 \\ \phi_2(\sigma) = 0, & \phi'_2(\sigma) = 0 \end{array}\right\}. \qquad \ldots \quad \ldots \quad \ldots (3.113)$$

The boundary condition $\phi_1(1) = 0$ gives

$$D_1' + D_3' + \frac{K_3'}{16} + \frac{K_1'^3}{256\times 86400} + \frac{K_1'^2}{256\times 72\times 640}\psi_1' + \frac{K_1'}{96\times 72\times 288}\psi_2'$$

$$+ \frac{\psi_4'}{384\times 96} - \frac{\psi_7'}{128} + 4\psi_{10}' = 0$$

or

$$D'_1 + D'_3 + \frac{1}{16}K'_3 = P' \text{ (say)}. \qquad \ldots \quad \ldots \quad \ldots (3.114)$$

The boundary condition $\phi_2(\sigma) = 0$ gives

$$D_1' + (D_2' \log\sigma + D_3')\sigma^2 + \frac{K_3'}{16}\sigma^4 + \frac{K_1'^3}{256\times 86400}\sigma^{12} + \frac{K_1'^2}{256\times 72\times 640}$$

$$\times(\psi_1' + \chi_2' \log\sigma)\sigma^{10} + \frac{K_1'}{96\times 72\times 288}\{\psi_2' + \psi_3'\log\sigma + \chi_5'(\log\sigma)^2\}\sigma^8 + \frac{1}{384\times 96}$$

$$\times\{\psi_4' + \psi_5'\log\sigma + \psi_6'(\log\sigma)^2 + \chi_9'(\log\sigma)^3\}\sigma^6 - \frac{1}{128}\{\psi_7' + \psi_8'\log\sigma + \psi_9'(\log\sigma)^2$$

$$+ \chi_{13}'(\log\sigma)^3\}\sigma^4 + \left[\{4 - 2\log\sigma + 2(\log\sigma)^2\}\psi_{10}' + \frac{1}{12}\chi_{15}'(\log\sigma)^3\right]\sigma^2 = 0$$

or

$$D'_1 + (D'_2\log\sigma + D'_3)\sigma^2 + \frac{K'_3}{16}\sigma^4 = Q' \text{ (say)}. \qquad \ldots \quad \ldots (3.115)$$

The boundary condition $\phi'_2(1) = 0$ gives

$$2D_3' + D_2' + \frac{K_3'}{4} + \frac{K_1'^3}{256\times 7200} + \frac{K_1'^2}{256\times 72\times 640}[10\psi_1' + \chi_2'] + \frac{K_1'}{96\times 72\times 288}$$

$$\times[8\psi_2' + \psi_3'] + \frac{1}{384\times 96}\{6\psi_4' + \psi_5'] - \frac{1}{128}[4\psi_7' + \psi_8'] + 6\psi_{10}' = 0$$

or

$$2D'_3 + D'_2 + \frac{K'_3}{4} = R' \text{ (say)} \qquad \ldots \quad \ldots \quad \ldots (3.116)$$

and the boundary condition $\phi'_2(\sigma) = 0$ gives

$$(2D_2' \log \sigma + 2D_3' + D_2')\sigma + \frac{K_3'}{4}\sigma^3 + \frac{K_1'^3}{256\times 7200}\sigma^{11} + \frac{K_1'^2}{256\times 72\times 640}[10\psi_1' + \chi_2'$$

$$+10\chi_2' \log \sigma]\sigma^9 + \frac{K_1'}{96\times 72\times 288}[8\psi_2' + \psi_3' + (8\psi_3' + 2\chi_5') \log \sigma + 8\chi_5' (\log \sigma)^2]\sigma^7$$

$$+\frac{1}{384\times 96}[6\psi_4' + \psi_5' + (6\psi_5' + 2\psi_6') \log \sigma + (6\psi_6' + 3\chi_9') (\log \sigma)^2 + 6\chi_9' (\log \sigma)^3]\sigma^5$$

$$-\frac{1}{128}[4\psi_7' + \psi_8' + (4\psi_8' + 2\psi_9') \log \sigma + (4\psi_9' + 3\chi_{13}') (\log \sigma)^2 + 4\chi_{13}' (\log \sigma)^3]\sigma^3$$

$$+\left[\{6 + 4 (\log \sigma)^2\}\psi_{10}' + \frac{1}{12}\chi_{15}'\{2 (\log \sigma)^3 + 3 (\log \sigma)^2\}\right]\sigma = 0,$$

or

$$(2D'_2 \log \sigma + 2D'_3 + D'_2)\sigma + \frac{K'_3}{4}\sigma^3 = S' \text{ (say)}. \quad \ldots \quad \ldots (3.117)$$

Solving (3.114) to (3.117), we get

$$D'_1 = \frac{2\sigma \log \sigma(2P' - R') + (S' - R'\sigma)}{4\sigma \log \sigma} + \frac{(\log \sigma + 1 - \sigma^2)K'_3}{16 \log \sigma} \quad \ldots \quad \ldots (3.118)$$

$$D'_2 = \frac{S' - R'\sigma}{2\sigma \log \sigma} + \frac{K'_3(1-\sigma^2)}{8 \log \sigma} \quad \ldots \quad \ldots (3.119)$$

$$D'_3 = \frac{R'\sigma(+2 \log \sigma) - S'}{4\sigma \log \sigma} - \frac{(2 \log \sigma + 1 - \sigma^2)}{16 \log \sigma}K'_3 \quad \ldots \quad \ldots (3.120)$$

and

$$K'_3 = \frac{4[4\sigma \log \sigma(Q' - P') + (1-\sigma^2)(\sigma R' - S') + 2\sigma \log \sigma(R' - \sigma S')]}{\sigma(1-\sigma^2)[(1-\sigma^2) + (1+\sigma^2) \log \sigma]}. \quad (3.121)$$

Hence the second order perturbation solution of $\phi(\eta)$ satisfying the boundary conditions (3.12) is

$$\phi(\eta) = \phi_0(\eta) + \lambda_2\phi_1(\eta) + \lambda_2^2\phi_2(\eta) \quad \ldots \quad \ldots \quad \ldots (3.122)$$

where $\phi_0(\eta)$, $\phi_1(\eta)$, and $\phi_2(\eta)$ are given by (3.68), (3.74) and (3.86) respectively.

For the solution of equation (2.6), we express a power series for small values of λ_1 and λ_2 near $\lambda_1 = 0 = \lambda_2$ as

$$F(\eta) = F_0(\eta) + (\lambda_1+\lambda_2)F_1(\eta) + (\lambda_1+\lambda_2)^2F_2(\eta) + \ldots + (\lambda_1+\lambda_2)^nF_n(\eta) + \ldots (3.123)$$

Substituting (3.1), (3.2) and (3.123) in eqn. (2.6) and equating like powers of λ_1 and λ_2, we get

$$F''_0 + \frac{1}{\eta}F'_0 = 0 \quad \ldots \quad \ldots \quad \ldots (3.124)$$

$$F''_1 + \frac{1}{\eta}F'_1 - \frac{1}{\eta}[F'_0f_0 - F_0f'_0] = 0 \quad \ldots \quad \ldots (3.125)$$

$$F''_1 + \frac{1}{\eta}F'_1 - \frac{1}{\eta}[F'_0\phi_0 - F_0\phi'_0] = 0 \quad \ldots \quad \ldots (3.126)$$

$$F''_2 + \frac{1}{\eta}F'_2 - \frac{1}{\eta}[F'_0f_1 + F'_1f_0 - F_0f'_1 - F_1f'_0] = 0 \quad \ldots (3.127)$$

$$F''_2 + \frac{1}{\eta}F'_2 - \frac{1}{\eta}[F'_0\phi_1 + F'_1\phi_0 - F_0\phi'_1 - F_1\phi'_0] = 0. \quad \ldots (3.128)$$

The boundary conditions to be satisfied by F_n's are

$$F_n(\sigma) = 0, \quad F_n(1) = 0. \qquad \text{(3.129)}$$

The solution of eqn. (2.6) using (3.124) to (3.129) is

$$F(\eta) = 0. \qquad \text{(3.130)}$$

Hence with the help of (2.1), (2.2), (3.67), (3.122) and (3.130), we get

$$v' = \frac{f(\eta)}{\eta}\lambda_1 + \frac{\phi(\eta)}{\eta}\lambda_2 \qquad \text{(3.131)}$$

$$u' = -\frac{\lambda_1}{\eta}f'(\eta)\bar{x} - \frac{\lambda_2}{\eta}\phi'(\eta)\bar{x}. \qquad \text{(3.132)}$$

Now substituting the values of v' and w_0 in (1.30) from (3.131) and (1.27) respectively, we get

$$\frac{d^2w'}{d\eta^2} + \frac{1}{\eta}\frac{dw'}{d\eta} - \frac{1}{\eta^2}w' - \frac{(f\lambda_1+\phi\lambda_2)}{\eta}\left(\frac{dw'}{d\eta}+\frac{w'}{\eta}\right) = \frac{2(\Omega_2-\Omega_1\sigma)}{\eta(1-\sigma^2)}(f\lambda_1+\phi\lambda_2). \qquad \text{(3.133)}$$

The solution of eqn. (3.133) is

$$w' = \frac{1}{\eta}\left[\int\left\{\frac{2(\Omega_2-\Omega_1\sigma)}{1-\sigma^2}\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}\,.\,e^{-\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta + C'\right\}e^{\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}\,.\,\eta d\eta + C''\right] \qquad \text{(3.134)}$$

where C' and C'' are constants of integration to be determined with the following boundary conditions:

$$w'(\sigma) = 0, \quad w'(1) = 0. \qquad \text{(3.135)}$$

By (3.134) and (3.135), constants C' and C'' are:

$$C' = \frac{\alpha(1)-\alpha(\sigma)}{\beta(\sigma)-\beta(1)} \qquad \text{(3.136)}$$

$$C'' = \frac{\alpha(\sigma)\,.\,\beta(1)-\alpha(1)\,.\,\beta(\sigma)}{\beta(\sigma)-\beta(1)} \qquad \text{(3.137)}$$

where $\alpha(\sigma)$, $\alpha(1)$, $\beta(\sigma)$ and $\beta(1)$ are

$$\alpha(\sigma) = \left[\int\left\{\frac{2(\Omega_2-\sigma\Omega_1)}{1-\sigma^2}\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}\,.\,e^{-\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta\right\}.\,\eta e^{\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta\right]_{\eta=\sigma} \qquad \text{(3.138)}$$

$$\alpha(1) = \left[\int\left\{\frac{2(\Omega_2-\sigma\Omega_1)}{1-\sigma^2}\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}\,.\,e^{-\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta\right\}\eta e^{\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta\right]_{\eta=1} \qquad \text{(3.139)}$$

$$\beta(\sigma) = \left[\int\eta e^{\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta\right]_{\eta=\sigma} \qquad \text{(3.140)}$$

$$\beta(1) = \left[\int\eta e^{\int\frac{(f\lambda_1+\phi\lambda_2)}{\eta}d\eta}d\eta\right]_{\eta=1}. \qquad \text{(3.141)}$$

Again the approximate solution for w', by neglecting the terms of second and higher order terms of λ_1 and λ_2, is

$$w' = \frac{1}{384}\left\{\frac{2(\Omega_2-\Omega_1\sigma)}{1-\sigma^2}+C'\right\}\{(A_1\lambda_1+A'_1\lambda_2)(192\eta\log\eta-96\eta)$$
$$+(A_3\lambda_1+A'_3\lambda_2)(48\eta^3\log\eta-36\eta^3)+48\eta^3(A_2\lambda_1+A'_2\lambda_2)$$
$$+(K_1\lambda_1+K'_1\lambda_2)\eta^5\}+\tfrac{1}{2}C'\eta+\frac{1}{\eta}C'' \quad \ldots \quad (3.142)$$

where C' and C'' are given by

$$C' = \frac{\bar{\alpha}(1)-\bar{\alpha}(\sigma)}{\bar{\beta}(\sigma)-\bar{\beta}(1)} \quad \ldots \quad (3.143)$$

and

$$C'' = \frac{\bar{\alpha}(\sigma)\,.\,\bar{\beta}(1)-\bar{\alpha}(1)\,.\,\bar{\beta}(\sigma)}{\bar{\beta}(\sigma)-\bar{\beta}(1)} \quad \ldots \quad (3.144)$$

where $\bar{\alpha}(\sigma)$, $\bar{\alpha}(1)$, $\bar{\beta}(\sigma)$ and $\bar{\beta}(1)$ are given by

$$\bar{\alpha}(\sigma) = \frac{2(\Omega_2-\Omega_1\sigma)}{384(1-\sigma^2)}\{(192\sigma^2\log\sigma-96\sigma^2)(A_1\lambda_1+A'_1\lambda_2)$$
$$+(48\sigma^4\log\sigma-36\sigma^4)(A_3\lambda_1+A'_3\lambda_2)+48\sigma^4(A_2\lambda_1+A'_2\lambda_2)+\sigma^6(K_1\lambda_1+K'_1\lambda_2)\} \quad \ldots \quad (3.145)$$

$$\bar{\alpha}(1) = \frac{-2(\Omega_2-\Omega_1\sigma)}{384(1-\sigma^2)}\{96(A_1\lambda_1+A'_1\lambda_2)+36(A_3\lambda_1+A'_3\lambda_2)$$
$$-48(A_2\lambda_1+A'_2\lambda_2)-(K_1\lambda_1+K'_1\lambda_2)\} \quad \ldots \quad (3.146)$$

$$\bar{\beta}(\sigma) = \frac{1}{384}\{192\sigma^2+(192\sigma^2\log\sigma-96\sigma^2)(A_1\lambda_1+A'_1\lambda_2)$$
$$+(48\sigma^4\log\sigma-36\sigma^4)(A_3\lambda_1+A'_3\lambda_2)+48\sigma^4(A_2\lambda_1+A'_2\lambda_2)+\sigma^6(K_1\lambda_1+K'_1\lambda_2)\} \quad \ldots \quad (3.147)$$

$$\bar{\beta}(1) = \frac{1}{384}\{192-96(A_1\lambda_1+A'_1\lambda_2)-36(A_3\lambda_1+A'_3\lambda_2)$$
$$+48(A_2\lambda_1+A'_2\lambda_2)+(K_1\lambda_1+K'_1\lambda_2)\} \quad \ldots \quad (3.148)$$

where A_1, A_2, A_3, K_1, A'_1, A'_2, A'_3 and K'_1 are given by (3.14), (3.15), (3.16), (3.17), (3.69), (3.70), (3.71) and (3.72) respectively.

Pressure Distribution

By substituting (2.6) and (2.8) in (2.3), we get

$$\frac{\partial p'}{\partial \bar{x}} = -\bar{x}K. \quad \ldots \quad (4.1)$$

By (2.4), we get

$$\frac{\partial p'}{\partial \eta} = \lambda_1\frac{(\eta f''-f')}{\eta^2}+\lambda_2\frac{(\eta\phi''-\phi')}{\eta^2}-\lambda_1^2\frac{(\eta ff'-f^2)}{\eta^3}-\frac{\lambda_2^2(\eta\phi\phi'-\phi^2)}{\eta^3}$$
$$+\frac{\lambda_1\lambda_2}{\eta^3}(\eta f\phi'-2f\phi+\eta\phi f')+\frac{2w_0w'}{\eta}+\frac{w'^2}{\eta}. \quad \ldots \quad (4.2)$$

By (1.23), we get

$$\frac{\partial p_0}{\partial \bar{x}} = 0 \quad \ldots \quad (4.3)$$

and by (1.24) and (1.27), we get

$$\frac{\partial p_0}{\partial \eta} = \frac{1}{\eta} \cdot \frac{1}{(1-\sigma^2)^2} \left[(\Omega_2 - \Omega_1\sigma)\eta + \frac{\sigma}{\eta}(\Omega_1 - \Omega_2\sigma) \right]^2 . \quad \ldots \quad (4.4)$$

Therefore, by (4.1) to (4.4) and (1.22), we get

$$\bar{p}(\bar{x}, \eta) = \frac{1}{(1-\sigma^2)^2} \left[(\Omega_2 - \Omega_1\sigma)^2 \frac{\eta^2}{2} - \sigma^2(\Omega_1 - \Omega_2\sigma)^2 \cdot \frac{1}{2\eta^2} + 2\sigma(\Omega_2 - \Omega_1\sigma) \times (\Omega_1 - \Omega_2\sigma) \log \eta \right] + \frac{\lambda_1 f'}{\eta} + \frac{\lambda_2 \phi'}{\eta} - \frac{\lambda_1{}^2 f^2}{2\eta^2} - \frac{\lambda_2{}^2 \phi^2}{2\eta^2} - \frac{\bar{x}^2 k}{2} + \int \left[\frac{\lambda_1 \lambda_2}{\eta^3} (\eta f \phi' - 2 f \phi + \eta \phi f') + \frac{2 w_0 w'}{\eta} + \frac{w'^2}{\eta} \right] d\eta + \text{const.} \quad \ldots \quad (4.5)$$

From (4.5), the pressure drop in axial direction is

$$\bar{p}(0, \eta) - \bar{p}(\bar{x}, \eta) = \frac{\bar{x}^2 K}{2} . \quad \ldots \quad (4.6)$$

Velocity Components

From (1.22), (1.27), (3.131), (3.132) and (3.142), we have

$$\bar{u} = -\frac{\lambda_1}{\eta} f'(\eta)\bar{x} - \frac{\lambda_2}{\eta} \phi'(\eta)\bar{x} \quad \ldots \quad (5.1)$$

$$\bar{v} = \frac{f(\eta)}{\eta} \lambda_1 + \frac{\phi(\eta)}{\eta} \lambda_2 \quad \ldots \quad (5.2)$$

$$\bar{w} = \frac{1}{(1-\sigma^2)} \left[(\Omega_2 - \Omega_1\sigma)\eta + \frac{\sigma}{\eta}(\Omega_1 - \Omega_2\sigma) \right] + \frac{1}{384} \left\{ \frac{2(\Omega_2 - \Omega_1\sigma)}{1-\sigma^2} + C' \right\} \times \{ (A_1\lambda_1 + A'_1\lambda_2)(192\eta \log \eta - 96\eta) + (A_3\lambda_1 + A'_3\lambda_2)(48\eta^3 \log \eta - 36\eta^3) + 48\eta^3(A_2\lambda_1 + A'_2\lambda_2) + (K_1\lambda_1 + K'_1\lambda_2)\eta^5 \} + \frac{C'\eta}{2} + \frac{C''}{\eta} . \quad \ldots \quad (5.3)$$

Particular Cases

(1) *The outer cylinder is non-porous, i.e.* $\lambda_1 = 0$—Putting $\lambda_1 = 0$ in eqns. (4.6), (5.1), (5.2) and (5.3), we get

$$\bar{p}(0, \eta) - \bar{p}(\bar{x}, \eta) = \frac{\bar{x}^2 K}{2} \quad \ldots \quad (6.1)$$

$$\bar{u} = -\frac{\lambda_2}{\eta} \phi'(\eta)\bar{x} \quad \ldots \quad (6.2)$$

$$\bar{v} = \frac{\phi(\eta)}{\eta} \lambda_2 \quad \ldots \quad (6.3)$$

$$\bar{w} = \frac{1}{(1-\sigma^2)}\left[(\Omega_2-\Omega_1\sigma)\eta+\frac{\sigma}{\eta}(\Omega_1-\Omega_2\sigma)\right]+\frac{\lambda_2}{384}\left\{\frac{2(\Omega_2-\Omega_1\sigma)}{1-\sigma^2}+C'\right\}$$

$$\times\{A'_1(192\eta\log\eta-36\eta)+A'_3(48\eta^3\log\eta-36\eta^3)+48A'_2\eta^3+K'_1\eta^5\}$$

$$+\tfrac{1}{2}C'\eta+\frac{C''}{\eta}. \quad \ldots \quad (6.4)$$

(2) *The inner cylinder is non-porous, i.e.* $\lambda_2 = 0$—Putting $\lambda_2 = 0$ in eqns. (4.6), (5.1), (5.2) and (5.3), we get

$$\bar{p}(0,\eta)-\bar{p}(\bar{x},\eta) = \frac{\bar{x}^2K}{2} \quad \ldots \quad (6.5)$$

$$\bar{u} = -\frac{\lambda_1}{\eta}f'(\eta)\bar{x} \quad \ldots \quad (6.6)$$

$$\bar{v} = \frac{f(\eta)}{\eta}\lambda_1 \quad \ldots \quad (6.7)$$

$$\bar{w} = \frac{1}{(1-\sigma^2)}\left[(\Omega_2-\Omega_1\sigma)\eta+\frac{\sigma}{\eta}(\Omega_1-\Omega_2\sigma)\right]+\frac{\lambda_1}{384}\left\{\frac{2(\Omega_2-\Omega_1\sigma)}{1-\sigma^2}+C'\right\}$$

$$\times\{A_1(192\eta\log\eta-36\eta)+A_3(48\eta^3\log\eta-36\eta^3)+48A_2\eta^3+K_1\eta^5\}$$

$$+\tfrac{1}{2}C'\eta+\frac{C''}{\eta}. \quad \ldots \quad (6.8)$$

(3) *When the cylinders have the same permeability, i.e.* $\lambda_1 = \lambda_2$ —Let $\lambda_1 = \lambda_2 = \lambda$ (say). Then putting $\lambda_1 = \lambda_2 = \lambda$ in eqns. (4.6), (5.1), (5.2) and (5.3), we get (neglecting the terms of λ^2 and so on)

$$\bar{p}(0,\eta)-\bar{p}(\bar{x},\eta) = \frac{\bar{x}^2K}{2} \quad \ldots \quad (6.9)$$

$$\bar{u} = \text{zero} \quad \ldots \quad (6.10)$$

$$\bar{v}\eta = \text{const.} \quad \ldots \quad (6.11)$$

$$\bar{w} = \frac{1}{(1-\sigma^2)}\left[(\Omega_2-\Omega_1\sigma)\eta+\frac{\sigma}{\eta}(\Omega_1-\Omega_2\sigma)\right]+\frac{\lambda}{384}\left\{\frac{2(\Omega_2-\Omega_1\sigma)}{1-\sigma^2}+C'\right\}$$

$$\times(192\eta\log\eta-96\eta)+\tfrac{1}{2}C'\eta+\frac{C''}{\eta}. \quad \ldots \quad (6.12)$$

Numerical Discussion and Graphs

The calculation for the longitudinal velocity profiles has been worked out from equation (5.1) for $\lambda_1 = 0{\cdot}1$ and $\lambda_2 = 0{\cdot}2$ for different values of $\bar{x}$ ($\bar{x} = 100, 200, 300, 400, 500$) and are shown in Fig. 1. In Fig. 2, the graphs for longitudinal velocity profiles have been drawn from eqn. (6.6) at $\bar{x} = 100$, for $\lambda_2 = 0$ and for different values of λ_1 ($\lambda_1 = 0{\cdot}1, 0{\cdot}2, 0{\cdot}3, 0{\cdot}4, 0{\cdot}5$). From the graphs it is evident that the longitudinal velocity profiles in both the cases shift towards the inner cylinder.

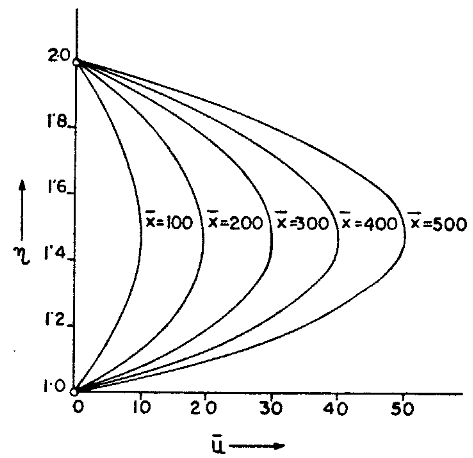

FIG. 1. Velocity profiles of longitudinal velocity plotted against η ($\lambda_1 = 0{\cdot}1$ and $\lambda_2 = 0{\cdot}2$).

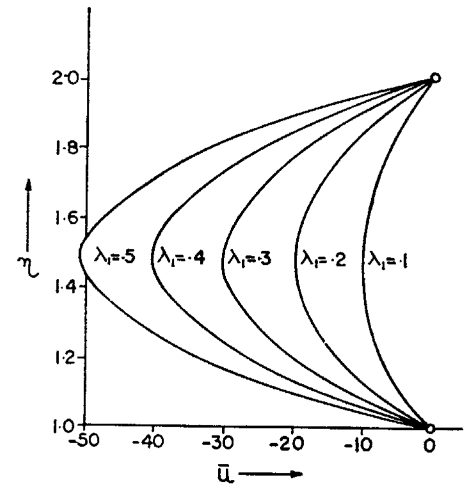

FIG. 2. Velocity profiles of longitudinal velocity plotted against η ($\lambda_2 = 0, \bar{x} = 100$).

The calculations for the transversal velocity profiles have been worked out from eqn. (6.6) in the case when the inner cylinder is non-porous, i.e. $\lambda_2 = 0$, and from eqn. (6.11) when both the cylinders have the same permeability, i.e. $\lambda_1 = \lambda_2 = \lambda$, and are shown in Figs. 3 and 4 respectively.

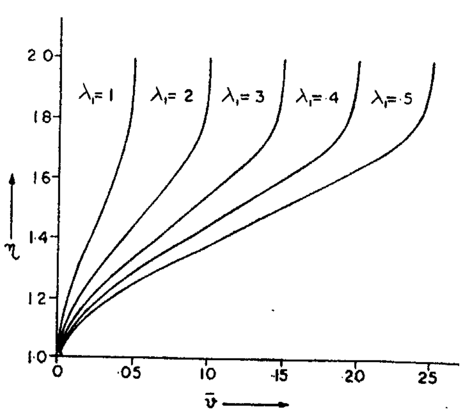

FIG. 3. Velocity profiles of transverse velocity plotted against η ($\lambda_2 = 0$).

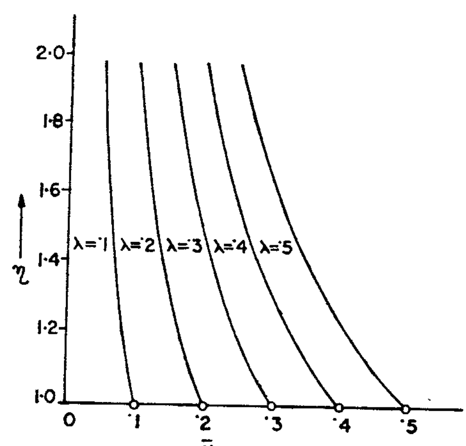

FIG. 4. Velocity profiles of transverse velocity plotted against η ($\lambda_1 = \lambda_2 = \lambda$).

The graphs for the azimuthal velocity profiles have been drawn for the case when the inner cylinder is rotating with twice the angular velocity of the outer cylinder in the same direction and have been shown in Figs. 5, 6 and 7 for different values of λ_1 and λ_2. It may, however, be seen that these velocity profiles always remain parabolic, for all the ratios of the angular velocities of the cylinders. Further the azimuthal velocity decreases with the increase of λ_1 and λ_2.

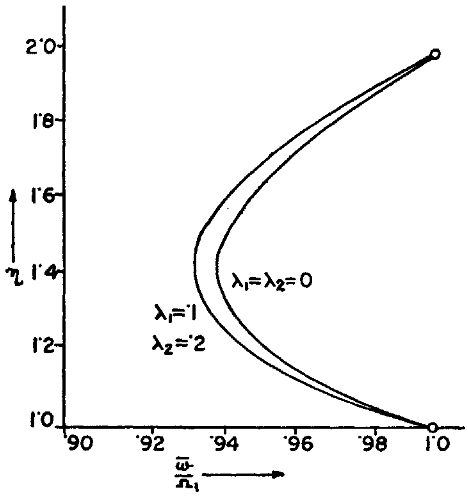

Fig. 5. Velocity profiles plotted for $\frac{\bar{w}}{\Omega_1}$ versus η.

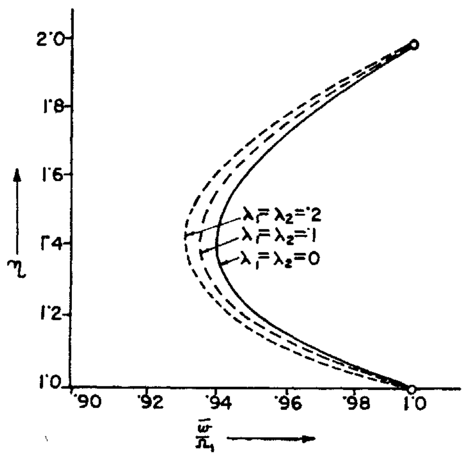

Fig. 6. Velocity profiles plotted for $\frac{\bar{w}}{\Omega_1}$ versus η ($\lambda_1 = \lambda_2$).

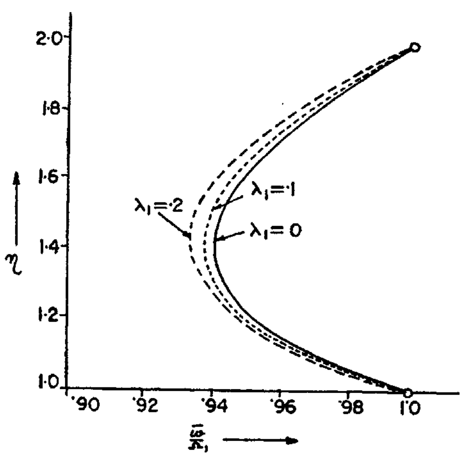

Fig. 7. Velocity profiles plotted for $\frac{\bar{w}}{\Omega_1}$ versus η ($\lambda_2 = 0$).

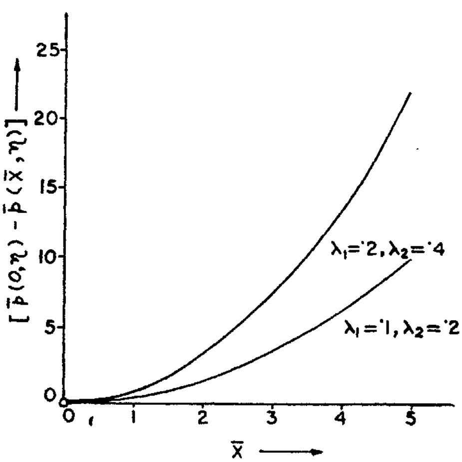

Fig. 8. Axial pressure drop versus $\bar{x}$.

Fig. 8 gives the drop in axial pressure for the case when the permeability of the inner cylinder is greater than that of the outer cylinder; it decreases with the increase of λ_1 and λ_2.

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