SELF-GRAVITATIONAL INSTABILITY OF A ROTATING FLUID WITH FINITE RESISTIVITY

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An attempt has been made to study the stability of a rotating self-gravitational fluid with finite resistivity and variable density. Through variational formulation of the problem it is found that the system cannot be overstable and if density varies exponentially it is conditionally stable.

1. Introduction

Hide (1955, 1956) and Chandrasekhar (1961) have made an extensive study about the role of dissipative processes in the investigation of instability of the fluid layer which reveals features both of heterogeneity and variation in density due to thermal expansion. Shafranov (1960) and Oganesyan (1960*a*, *b*) have explained the importance of such a study in astronomical context. Hide (1955) has shown that the stability characteristics are dependent on three adjustable parameters incorporating the roles of viscosity, resistivity and buoyancy forces. The discussion involves only the cases of zero and infinite resistivity. In the case of zero resistivity instability exists for all positive values of parameter characterizing the buoyancy forces while the negativeness of the same parameter leads to stable equilibrium. Using energy method Vardanian and Oganesyan (1962) have extended Hide's analysis for a simple model in infinitely conducting self-gravitational fluid. The fluid is stable for a critical wavenumber in the presence of magnetic field and conditions under which magnetic field suppresses the development of unstable harmonics have also been obtained. Jukes (1963) has discussed the effect of finite resistivity in a different context. Sundaram (1968) has analysed the simultaneous roles of finite resistivity and viscosity in a fluid layer of variable density and has established that the system is unstable for all wavenumbers and finite resistivity.

We have studied the effect of uniform rotation on the stability of a magnetized self-gravitational fluid with viscosity and finite resistivity. Conditions connecting different parameters are derived for the stability of the system and discussion is also extended for some limiting cases.

2. Formulation of the Problem

We consider in equilibrium, a horizontal strata of heavy, viscous, incompressible, conducting fluid of variable density. The fluid is assumed to be self-gravitating in the presence of uniform rotation $\vec{\Omega}$ and magnetic field $\vec{H}$ both being parallel to the vertical coordinate z only. The pressure p_0, density ρ_0, coefficient of viscosity μ_0 and gravitational potential ϕ_0 are taken to be functions of z.

If p, ρ, ϕ, $\vec{V}(u, v, w)$ and $\vec{h}$ are small perturbations in pressure, density, potential, velocity and magnetic field then the relevant equations of the problem are

$$\frac{\partial \rho}{\partial t}+\left(\vec{V} \cdot \nabla\right)\rho_0=0 \qquad (2.1)$$

$$\nabla \cdot \vec{V}=0 \qquad (2.2)$$

$$\rho_0 \frac{\partial \vec{V}}{\partial t}=-\nabla p+\rho_0 \nabla \phi+\rho \nabla \phi_0+\nabla \cdot \left(\mu_0 \nabla \vec{V}\right) +(\nabla \mu_0 \cdot \nabla)\vec{V}+2\rho_0\left(\vec{V}\times\vec{\Omega}\right) +\frac{\mu}{4\pi}\left[\left(\vec{H} \cdot \nabla\right)\vec{h}-\nabla\left(\vec{H} \cdot \vec{h}\right)\right] \qquad (2.3)$$

$$\left(\frac{\partial}{\partial t}-\eta \nabla^2\right)\vec{h}=\left(\vec{H} \cdot \nabla\right)\vec{V} \qquad (2.4)$$

$$\nabla \cdot \vec{h}=0 \qquad (2.5)$$

$$\nabla^2 \phi=-4\pi G\rho \qquad (2.6)$$

where μ is the magnetic permeability, η the resistivity and G is the gravitational constant.

Choosing dependence of the perturbation of the form

$$q(z) \exp(ik_x x+ik_y y+nt),$$

where $k^2=k_x^2+k_y^2$ and denoting $D=\partial/\partial z$ the set of eqns. (2.1)–(2.6) gives

$$\rho=-\frac{w}{n} D\rho_0 \qquad (2.7)$$

$$k_x u+k_y v=i\, Dw \qquad (2.8)$$

$$(n\rho_0+\mu_0 k^2)\zeta-D(\mu_0 D\zeta)-\frac{\mu H}{4\pi} D\xi=2\Omega\rho_0\, Dw \qquad (2.9)$$

$$n\rho_0\, Dw=-k^2(p-\rho_0\phi)+\mu_0 \nabla^2(Dw)+D\mu_0(D^2+k^2)w+\frac{\mu H}{4\pi} \nabla^2 h_z-2\Omega\rho_0\zeta \qquad (2.10)$$

$$Dp=-\left[n\rho_0+\frac{(D\rho_0)(D\phi_0)}{n}-\mu_0 \nabla^2\right]w+\rho_0(D\phi)+2(D\mu_0)(Dw) \qquad (2.11)$$

$$(n-\eta \nabla^2)h_z=H\, Dw \qquad (2.12)$$

$$k_x h_x+k_y h_y=i\, Dh_z \qquad (2.13)$$

$$\nabla^2 \phi=\frac{4\pi G\, w(D\rho_0)}{n} \qquad (2.14)$$

where

$$\zeta=ik_x v-ik_y u, \quad \xi=ik_x h_y-ik_y h_x. \qquad (2.15)$$

Taking curl of eqn. (2.4) and writing z-component, we get

$$(n-\eta\nabla^2)\xi = HD\zeta. \quad .. \quad .. \quad .. \quad (2.16)$$

Equation (2.9) is obtained on subtracting k_y times the z-component from k_x times the y-component of eqn. (2.3). Similarly eqn. (2.10) is obtained on adding k_x times the x-component into k_y times the y-component of eqn. (2.3). In deducing eqns. (2.9) and (2.10) we have also made use of eqns. (2.8) and (2.13). With the help of eqn. (2.7), eqn. (2.6) yields eqn. (2.14).

Eliminating p between eqns. (2.10) and (2.11) we obtain

$$\left[n\rho_0\nabla^2-\mu_0\nabla^4+n(D\rho_0)D-\frac{k^2}{n}(D\rho_0)(D\phi_0)\right]w$$
$$-2(D\mu_0)\nabla^2(Dw)-(D^2\mu_0)(D^2+k^2)w-k^2(D\rho_0)\phi$$
$$-\frac{\mu H}{4\pi}\nabla^2 Dh_z+2\Omega\, D(\rho_0\zeta) = 0. \quad .. \quad .. \quad .. \quad (2.17)$$

Equations (2.9), (2.12), (2.14) and (2.17) are the basic equations for further analysis.

3. Boundary Conditions

Assuming that the fluid is confined between the planes $z = 0$ and $z = d$, the boundary conditions to be satisfied (Hide 1955) by the velocity and magnetic field are

$$w = D^2w = Dh_z = 0 \quad \text{(free surface)} \quad .. \quad .. \quad (3.1)$$

$$w = Dw = h_z = D^2h_z = 0 \quad \text{(rigid surface)} \quad .. \quad .. \quad (3.2)$$

and according to Chandrasekhar (1961; p. 22, 163)

$$\xi = D\zeta = 0 \quad \text{(free surface)} \quad .. \quad .. \quad .. \quad (3.3)$$

$$\zeta = D\xi = 0 \quad \text{(rigid surface).} \quad .. \quad .. \quad .. \quad (3.4)$$

In view of the fact that ϕ and the normal component of grad ϕ are continuous across the surface and on matching the solutions at the boundaries $z = 0$ and $z = d$ we obtain (Sundaram 1968)

$$(D-k)\phi = 0 \ \text{at}\ z = 0 \quad .. \quad .. \quad .. \quad .. \quad (3.5)$$

$$(D+k)\phi = 0 \ \text{at}\ z = d. \quad .. \quad .. \quad .. \quad (3.6)$$

4. A Variational Procedure

For variational formulation of the problem, let x_i and x_j be two characteristic values and distinguish the solutions belonging to these by the subscripts i and j. These characteristic values are due to the requirement that the solution of eqns. (2.9), (2.12), (2.14) and (2.17) satisfies the boundary conditions (3.1)–(3.6).

We consider eqn. (2.17) for the characteristic value n_i and multiply it by w_j. Making use of eqns. (2.12) and (2.14) and the boundary conditions, we integrate between the limit 0 and d and obtain

$$n_i \int_0^d \rho_0 \, Dw_i Dw_j \, dz + n_i k^2 \int_0^d \rho_0 \, w_i w_j \, dz + \frac{k^2}{n_i} \int_0^d D\rho_0 D\phi_0 \, w_i w_j \, dz$$
$$+ \int_0^d \mu_0 [(D^2+k^2) w_i (D^2+k^2) w_j + 4k^2 Dw_i Dw_j] \, dz$$
$$+ \frac{\mu\eta}{4\pi} \int_0^d (D^2 h_i D^2 h_j + 2k^2 Dh_i Dh_j + k^4 h_i h_j) \, dz + \frac{\mu}{4\pi} n_j \int_0^d (Dh_i Dh_j + k^2 h_i h_j) \, dz$$
$$- \frac{n_j k^2}{4\pi G} \int_0^d (D\phi_i D\phi_j + k^2 \phi_i \phi_j) \, dz + 2\Omega \int_0^d \rho_0 \, Dw_j \zeta_i \, dz = 0. \quad \ldots \quad (4.1)$$

Using eqns. (2.9) and (2.16) and boundary conditions (3.3), (3.4) we evaluate the last integral of eqn. (4.1) in terms of integrals symmetric in the indices i and j and finally eqn. (4.1) comes out to be

$$n_i \left(I_1 + I_2 + \frac{I_3}{n_i^2} \right) + n_j (I_4 + I_5 - I_6) + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} = 0 \quad \ldots \quad (4.2)$$

where

$$\left.\begin{aligned}
I_1 &= \int_0^d \rho_0 (Dw_i Dw_j + k^2 w_i w_j) \, dz \\
I_2 &= \frac{\mu}{4\pi} \int_0^d \xi_i \xi_j \, dz \\
I_3 &= k^2 \int_0^d (D\rho_0)(D\phi_0) w_i w_j \, dz \\
I_4 &= \int_0^d \rho_0 \, \zeta_i \zeta_j \, dz \\
I_5 &= \frac{\mu}{4\pi} \int_0^d (Dh_i Dh_j + k^2 h_i h_j) \, dz \\
I_6 &= \frac{k^2}{4\pi G} \int_0^d (D\phi_i D\phi_j + k^2 \phi_i \phi_j) \, dz \\
I_7 &= \int_0^d \mu_0 [(D^2+k^2) w_i (D^2+k^2) w_j + 4k^2 Dw_i Dw_j] \, dz \\
I_8 &= \frac{\mu\eta}{4\pi} \int_0^d (D^2 h_i D^2 h_j + 2k^2 Dh_i Dh_j + k^4 h_i h_j) \, dz \\
I_9 &= k^2 \int_0^d \mu_0 \, \zeta_i \zeta_j \, dz \\
I_{10} &= \int_0^d \mu_0 \, D\zeta_i D\zeta_j \, dz \\
I_{11} &= \eta k^2 I_2 \\
I_{12} &= \frac{\mu\eta}{4\pi} \int_0^d D\xi_i D\xi_j \, dz.
\end{aligned}\right\} \quad \ldots \quad (4.3)$$

Writing $i = j$ in eqn. (4.2) and suppressing the subscripts we find

$$n\left(I_1+I_2+\frac{I_3}{n^2}+I_4+I_5-I_6\right)+I_7+I_8+I_9+I_{10}+I_{11}+I_{12} = 0 \quad .. \quad (4.4)$$

where the integrals I's have the same meaning with suppressed subscripts in the set (4.3). To see that eqn. (4.4) forms the basis for variational formulation we consider arbitrary variations δw, δh, $\delta\zeta$, $\delta\xi$ and $\delta\phi$ in w, h, ζ, ξ and ϕ respectively compatible with the boundary conditions (3.1)–(3.6). Consequently a variation δn in n and δI in I satisfy to the first order the relation

$$\delta n\left(I_1+I_2-\frac{I_3}{n^2}+I_4+I_5-I_6\right)+n\left(\delta I_1+\delta I_2+\frac{\delta I_3}{n^2}+\delta I_4+\delta I_5-\delta I_6\right)$$
$$+\delta I_7+\delta I_8+\delta I_9+\delta I_{10}+\delta I_{11}+\delta I_{12} = 0 \quad .. \quad .. \quad .. \quad (4.5)$$

where δI's are derived in accordance with expression given by the set (4.3) compatible with the boundary conditions and are given by

$$\left.\begin{aligned}
\delta I_1 &= 2\int_0^d \delta w[\rho_0 k^2 w - D(\rho_0 Dw)]\,dz \\
\delta I_2 &= \frac{\mu}{2\pi}\int_0^d \xi\,\delta\xi\,dz \\
\delta I_3 &= 2k^2\int_0^d (D\rho_0)(D\phi_0)w\,\delta w\,dz \\
\delta I_4 &= 2\int_0^d \rho_0\zeta\,\delta\zeta\,dz \\
\delta I_5 &= -\frac{\mu}{2\pi}\int_0^d \delta h\,\nabla^2 h\,dz \\
\delta I_6 &= -\frac{k^2}{2\pi G}\int_0^d \delta\phi\,\nabla^2\phi\,dz \\
\delta I_7 &= 2\int_0^d \delta w[D^2\{\mu_0(D^2+k^2)w\}+\mu_0 k^2(D^2+k^2)w-4k^2 D(\mu_0 Dw)]\,dz \\
\delta I_8 &= \frac{\mu\eta}{2\pi}\int_0^d \delta h\,\nabla^4 h\,dz \\
\delta I_9 &= 2k^2\int_0^d \mu_0\zeta\,\delta\zeta\,dz \\
\delta I_{10} &= -2\int_0^d \delta\zeta\,D(\mu_0 D\zeta)\,dz \\
\delta I_{11} &= \eta k^2\,\delta I_2 \\
\delta I_{12} &= -\frac{\mu\eta}{2\pi}\int_0^d \delta\xi\,D^2\xi\,dz.
\end{aligned}\right\} \quad (4.6)$$

Using eqn. (2.17) and equations obtained from eqns. (2.9), (2.12), (2.14) and (2.16) by giving arbitrary variations, it can be shown that

$$n\left(\delta I_1+\delta I_2+\frac{\delta I_3}{n^2}+\delta I_4+\delta I_5-\delta I_6\right)+\delta I_7+\delta I_8 \\ +\delta I_9+\delta I_{10}+\delta I_{11}+\delta I_{12} = 2\delta n(I_6-I_2-I_4-I_5). \quad .. \quad (4.7)$$

Combining eqns. (4.5) and (4.7) we get

$$\delta n\left(I_1-I_2-\frac{I_3}{n^2}-I_4-I_5+I_6\right) = 0.$$

Since the expression within the brackets does not vanish, in general, we infer that $\delta n = 0$. Therefore, a necessary and sufficient condition for δn to vanish to the first order for all small arbitrary variations is that the solution of eqns. (2.9), (2.12), (2.14) and (2.17) is the solution of the characteristic value problem.

If n_i is complex, we suppose that n_i and n_j are complex conjugates and from eqn. (4.2) we deduce that

$$\text{Re}\,(n) = -\frac{I_7+I_8+I_9+I_{10}+I_{11}+I_{12}}{I_1+I_2+\frac{I_3}{|n|^2}+I_4+I_5-I_6} \quad .. \quad .. \quad (4.8)$$

$$\text{Im}\,(n)\left(I_1+I_2-\frac{I_3}{|n|^2}-I_4-I_5+I_6\right) = 0. \quad .. \quad .. \quad (4.9)$$

As n is complex, Im $(n) \neq 0$, from eqn. (4.9) we have

$$I_1+I_2+I_6 = \frac{I_3}{|n|^2}+I_4+I_5. \quad .. \quad .. \quad .. \quad (4.10)$$

In view of the relation (4.10), eqn. (4.8) simplifies to

$$\text{Re}\,(n) = -\frac{I_7+I_8+I_9+I_{10}+I_{11}+I_{12}}{2(I_1+I_2)}. \quad .. \quad .. \quad (4.11)$$

From eqn. (4.11) we conclude that if n is complex Re $(n) < 0$ implying that if oscillatory modes exist they are stable, conversely overstability cannot occur.

5. Stratified Layer of Gravitating Fluid

As an illustration of the variational approach we consider a stratified fluid bounded by free surfaces and assume that density varies exponentially as the distance in vertical direction. Using subscript c for a constant quantity and denoting the kinematic viscosity by ν we write

$$\rho_0 = \rho_c\, e^{\beta z}, \quad \mu_0 = \nu_c \rho_c\, e^{\beta z} \quad .. \quad .. \quad .. \quad (5.1)$$

and therefore

$$\phi_0 = \frac{4\pi G\rho_c}{\beta^2}(1+\beta z-e^{\beta z}) \quad .. \quad .. \quad .. \quad (5.2)$$

where β is a constant.

Consistent with the boundary conditions (3.1)–(3.6) we let the trial solution for w, h, ϕ, ξ and ζ be

$$\left.\begin{array}{l} w = A \sin \alpha z, \;\; h = B \cos \alpha z, \\ (D \mp k)\phi = C e^{\beta z} \sin \alpha z, \;\; \xi = D \sin \alpha z, \;\; \zeta = E \cos \alpha z \end{array}\right\} \qquad (5.3)$$

where A, B, C, D and E are arbitrary constants and $\alpha = m\pi/d$, d being the dimension of the fluid measured along the z-axis and m is a positive integer. In the trial solution for ϕ the minus and plus signs correspond to the boundaries $z = 0$ and $z = d$ respectively.

On incorporating (5.1)–(5.3) in eqns. (2.9), (2.12), (2.16) and (2.17) we deduce the arbitrary constants. Dropping the subscript c these are expressed as

$$\left.\begin{array}{l} B = \dfrac{(H/\nu\alpha)A}{p+\epsilon t} \\[2ex] C = -\dfrac{\nu\alpha^2[1+(a-x)^2][(p+t)(p+2t)+\gamma^2]A}{x^2(p+t)} \\[2ex] D = -\dfrac{(\gamma/\nu)HA}{(p+t)(p+\epsilon t)} \\[2ex] E = \dfrac{\gamma\alpha A}{p+t} \end{array}\right\} \qquad (5.4)$$

where we have introduced the non-dimensional parameters

$$\left.\begin{array}{l} x = \dfrac{k}{\alpha}, \;\; t = 1+x^2, \;\; a = \dfrac{\beta}{\alpha}, \;\; \epsilon = \dfrac{\eta}{\nu}, \;\; p = \dfrac{n}{\nu\alpha^2}, \\[2ex] \gamma = \dfrac{2\Omega}{\nu\alpha^2}, \;\; \lambda^2 = \dfrac{\pi G\rho}{\nu^2\alpha^4}, \;\; \theta^2 = \dfrac{\mu H^2/4\pi\rho}{\alpha^2\nu^2}. \end{array}\right\} \qquad (5.5)$$

In the derivation of C it is found that $\phi_{z=0} = \phi_{z=d}$ which requires

$$e^{m\pi a} - 1 = \frac{4ax}{1+(a-x)^2} = X \qquad (5.6)$$

and we have made use of eqn. (5.6) while determining the constant C and integral I_6.

We calculate the integrals (4.3) and substitute their values in eqn. (4.4). Using the set (5.4) to eliminate arbitrary constants we obtain the dispersion relation

$$\begin{aligned} & ax(3+y)[p+t(\epsilon+6)]p^6 + [L + 6ax\,\epsilon t^2(3+y)]p^5 \\ & \quad + [\epsilon t\, L + 6ax\, t(3+y)(2t^2+\gamma^2) - 4yX\,\lambda^2\{3t^2+a^2(3t-2)\}]p^4 \\ & \quad + [S + at(3+y)\{6x\,\epsilon t(2t^2+\gamma^2) - m\pi y\,\lambda^2\theta^2\} - 2yX\,\lambda^2\{\gamma^2(y+1) \\ & \quad + 2\epsilon t[3t^2+a^2(3t-2)]\}]p^3 + [\epsilon t\, S - 2X\,\lambda^2\{2Mt\,\lambda^2x^2 \\ & \quad + 2yt^2(t^2+2a^2x^2) + y\gamma^2[2+a^2(\epsilon t+x^2) - 2m\pi ay\,\lambda^2t^2\theta^2(3+y)]p^2 \\ & \quad - \lambda^2[2Xt^2\{M\,\lambda^2x^2(2\epsilon+1) + 2y\,\epsilon t(t^2+a^2x^2)\} \\ & \quad + m\pi ay\,\theta^2(3+y)(t^3+\gamma^2) + 2yX\,\gamma^2\{2+x^2(y+1)\}]p \\ & \quad - 2M\epsilon\, Xx^2t^3\lambda^4 = 0 \end{aligned} \qquad (5.7)$$

where

$$\left.\begin{aligned} L &= ax(3+y)(13t^2+2\gamma^2)-2yX\,\lambda^2(2t+y-1) \\ S &= ax(3+y)(2t^2+\gamma^2)^2-2X\,\lambda^2[M\,\lambda^2x^2+yt\{6t^2+a^2(9t-8)\}] \\ M &= 8y-(3+y)(X+2),\ \ y = 1+a^2. \end{aligned}\right\} \quad .. \quad (5.8)$$

Writing eqn. (5.7) as

$$A_1p^7+A_2p^6+A_3p^5+A_4p^4+A_5p^3+A_6p^2+A_7p+A_8 = 0$$

we observe that coefficients A_1, A_2 and A_5 are positive definite. The coefficients A_3, A_4 and A_5, A_6, A_7 are respectively positive definite if

$$13(\epsilon+1)+2(\epsilon+3)\gamma^2 \geqslant 3a\,\lambda^2(2t+a^2) \quad .. \quad .. \quad .. \quad .. \quad (5.9)$$

$$24ax^3\,\lambda^4(2\epsilon+1)+2x^2(3+y)(2+\gamma^2)[2(3\epsilon+1)+\gamma^2]$$
$$\geqslant 48ax\,\lambda^2[4x^2\lambda^2+\epsilon t(2a^2x^2+t^2)]+3m\pi at\,\lambda^2\theta^2(3+y)[1+(a-x)^2]\,. \quad .. \quad (5.10)$$

From the inequalities (5.9) and (5.10) it follows that all the coefficients of eqn. (5.7) are positive and hence it will not admit any positive root. The roots will be negative or complex. If the roots are negative the perturbations are damped. If the roots are complex then as proved in § 4 the real part will be negative. This ensures the stability of the system.

We now investigate some limiting cases. The discussion is similar to that adopted for the general dispersion relation (5.7).

Case I: $a \ll 1,\ x \to 0$

The dispersion relation (5.7) reduces to the form

$$p^4+(\epsilon+3)p^3+(3\epsilon+3+\theta^2+\gamma^2)p^2+[3\epsilon+1+2\theta^2+(\epsilon+1)\gamma^2]p+(\epsilon+\theta^2)(1+\gamma^2) = 0,$$

which establishes that the system is stable.

Case II: $a \ll 1,\ x \gg 1$

Regarding $1/x^2$ to be a first order quantity it is found that an unstable mode exists and

$$p = \frac{\lambda}{2}\left(\frac{m\pi\epsilon x}{3\epsilon+1}\right)^{\frac{1}{2}}.$$

Case III: $a \gg 1$

The dispersion relation is obtained by retaining terms of second order in $1/a$ and it comes out to be

$$2\lambda^2(X+1)p^4+2\lambda^2(X+1)(2+\epsilon t)p^3+[2\lambda^2(X+1)\{2\gamma^2+t(3t-4)\}$$
$$+m\pi at\,\lambda^2\theta^2]p^2+2[t^2\lambda^2(X+1)\{\epsilon t+4x^2(2\epsilon+1)\}+2\gamma^2\lambda^2(X+1)(\epsilon t+x^2)$$
$$+m\pi at^2\,\lambda^2\theta^2]p+\lambda^2[2\epsilon t\,x^2(X+1)(4t^2+\gamma^2)+m\pi a\,\theta^2(t^3+\gamma^2)] = 0.$$

Since all the coefficients, except that of p^2, are positive real and from the coefficient of p^2 we conclude that the system is stable if x or $\gamma \geqslant (1/2)^{\frac{1}{2}}$. In case x, $\gamma < (1/2)^{\frac{1}{2}}$, the system is stable if $x \geqslant x_*$, where x_* is the root of the equation

$$3x^4+2x^2+2\gamma^2-1 = 0.$$

Case IV: $x \to \infty$

The fluid layer is unstable and p is given by

$$p = \frac{y}{x}\frac{X\lambda^2}{a(3+y)}.$$

Case V: $\gamma \gg 1$

Taking $1/\gamma^2$ to be a first order quantity the dispersion relation (5.7) reduces to the form

$$2ax(3+y)p^4+2ax\,t(3+y)(3+\epsilon)p^3+[ax(3+y)\{2t^2(3\epsilon+2)+\gamma^2\}$$
$$-2yX\,\lambda^2(1+y)]p^2+[ax\,\epsilon t(3+y)-2yX\,\lambda^2\{2+(1+y)(\epsilon t+x^2)\}]p$$
$$-y\lambda^2[m\pi a\,\theta^2(3+y)+2\epsilon t\,X(a^2x^2+2t)] = 0. \quad .. \quad .. \quad .. \quad (5.11)$$

We express eqn. (5.11) as

$$A_1p^4+A_2p^3+A_3p^2+A_4p+A_5 = 0.$$

In eqn. (5.11) the coefficients A_1 and A_2 are positive real and A_5 is negative real. The coefficient A_3 is positive real as it is containing γ^2. Therefore, irrespective of the sign of coefficient A_4 there is only one change of sign which ensure the existence of an unstable mode.

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