GENERATING FUNCTIONS FOR JACOBI, LAGUERRE AND BESSEL POLYNOMIALS

by I. A. KHAN, *Department of Mathematics, Govt. Engineering College, Rewa (M.P.)*

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The object of this paper is to obtain the linear generating functions for Jacobi, Laguerre and Bessel polynomials, involving the hypergeometric functions of two variables and three variables. The particular cases of our formulae lead to certain generalizations of the known and unknown results.

§ 1. By the use of Legendre transform, Tranter (1950) solved a boundary value problem proposed by Nicholson who failed to solve it by Bessel-Fourier analysis. Scott (1953) introduced a more general Jacobi transform and applied it to a problem on heat conduction. Moreover, the exhaustive work of Watson (1944) has shown a detailed treatment of Bessel functions, in various types of problems, and later on, the Bessel polynomials defined by Krall and Frink (1949) were related to these Bessel functions.

It is the usefulness of the Jacobi, Legendre, Bessel and other polynomials in applied field which makes the study of polynomial sets interesting. Obviously the generating functions always play a large and important role in the study of polynomial sets. Therefore, we establish here some generating relations for Jacobi, Laguerre and Bessel polynomials, involving Horn's functions of two variables (Erdélyi 1953, vol. I, pp. 224–26) and hypergeometric functions of three variables defined by Saran (1954) and Jain (1966).

Our formula (2.3) generalizes the well-known results due to Burchnall (1951) and Krall and Frink (1949). Moreover, the formula (4.6) generalizes the known result (Erdélyi 1953, vol. II, p. 189) (19)). Further, on specializing the parameters, some of our results yield the other interesting generating relations for Jacobi, Laguerre and Bessel polynomials.

The Jacobi polynomials (Rainville 1960, p. 254) are given by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1(-n,\, 1+\alpha+\beta+n;\ 1+\alpha;\ (1-x)/2), \quad \text{Re}\,(\alpha) > -1,\ \text{Re}\,(\beta) > -1. \qquad (1.1)$$

and the Laguerre polynomials (Rainville 1960, p. 200) are defined by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n;\ 1+\alpha;\ x), \quad \text{Re}\,(\alpha) > -1. \qquad (1.2)$$

The standard Bessel polynomials (Rainville 1960, p. 294) are defined by

$$\psi_n(c, x) = \frac{(c)_n}{n!}\, {}_2F_0(-n, c+n\,;\ -\,;\ x) \qquad \ldots \quad \ldots \quad (1.3)$$

which, on replacing x by $-x/b$, putting $c = a-1$ and multiplying by $n!/(a-1)_n$, reduces to Krall-Frink generalized Bessel polynomials defined by Krall and Frink (1949),

$$y_n(a, b, x) = {}_2F_0\left(-n, a-1+n\,;\ -\,;\ -\frac{x}{b}\right). \qquad \ldots \quad \ldots \quad (1.4)$$

The required Horn's functions H_3, Φ_1, Φ_3 and H_6 (Erdélyi 1953, vol. I, pp. 225–26) are defined by

$$H_3(\alpha, \beta, \gamma, x, y) = \sum_{m,\, n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n, \qquad \ldots \quad \ldots \quad (1.5)$$

$$|x| < r,\ |y| < s,\ r+(s-\tfrac{1}{2})^2 = \tfrac{1}{4}.$$

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,\, n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n, \qquad \ldots \quad \ldots \quad (1.6)$$

$$|x| < 1.$$

$$\Phi_3(\beta, \gamma, x, y) = \sum_{m,\, n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n \qquad \ldots \quad \ldots \quad \ldots \quad (1.7)$$

$$H_6(\alpha, \gamma, x, y) = \sum_{m,\, n=0}^{\infty} \frac{(\alpha)_{2m+n}}{(\gamma)_{m+n}\, m!\, n!} x^m y^n, \qquad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

$$|x| < \tfrac{1}{4}.$$

The required hypergeometric functions of three variables ${}_3\Phi_A^{(2)}$, ${}_3\Phi_M^{(4)}$, ${}_3\Phi_G^{(1)}$ defined by Jain (1966) and F_E defined by Saran (1954) are as follows:

$${}_3\Phi_A^{(2)}(a, b_1; c, c_2, c_3; x, y, z) = \sum_{m,\, n,\, p=0}^{\infty} \frac{(a)_{m+n+1p}(b_1)_m}{m!\, n!\, p!\, (c_1)_m (c_2)_n (c_3)_p} x^m y^n z^p, \qquad \ldots \quad (1.9)$$

$$|x| < 1.$$

$${}_3\Phi_M^{(4)}(a_1, b_1, b_1; c_1, c_2, c_2; x, y, z) = \sum_{m,\, n,\, p=0}^{\infty} \frac{(a_1)_m (b_1)_{m+p}}{(c_1)_m (c_2)_{n+p}\, m!\, n!\, p!} x^m y^n z^p, \qquad \ldots \quad (1.10)$$

$$|x| < 1.$$

$${}_3\Phi_G^{(1)}(a_1, a_1, a_1, b_1, b_2; c_1, c_2, c_2; x, y, z) = \sum_{m,\, n,\, p=0}^{\infty} \frac{(a_1)_{m+n+p}(b_1)_m (b_2)_n}{m!\, n!\, p!\, (c_1)_m (c_2)_{n+p}} x^m y^n z^p, \quad (1.11)$$

$$|x| < 1,\ |y| < 1.$$

$$F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \sum_{m,\, n,\, p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m (\beta_2)_{n+p}}{m!\, n!\, p!\, (\gamma_1)_m (\gamma_2)_n (\gamma_3)_p} x^m y^n z^p,$$

$$|x| < r,\ |y| < s,\ |z| < t,\ r+(\sqrt{s}+\sqrt{t})^2 = 1. \qquad (1.12)$$

§ 2. We prove here the following formulae:

$$H_3\left(1+\alpha+\beta,\ -\beta,\ \gamma,\ \frac{(x-1)y}{2},\ -y\right) = \sum_{m=0}^{\infty} \frac{(1+\alpha+\beta)_m}{(\gamma)_m} P_m^{(\beta-m,\ \alpha+m)}(x)\,.\,y^m, \tag{2.1}$$

$$\left|\frac{(x-1)y}{2}\right| < r,\ |y| < s,\ r+(s-\tfrac{1}{2})^2 = \tfrac{1}{4},\ \mathrm{Re}\,(\beta) > -1,\ \mathrm{Re}\,(\alpha) > -1.$$

$$H_6(-\alpha,\ \gamma,\ x,\ xy) = \sum_{n=0}^{\infty} \frac{(-\alpha)_{2n}}{(\gamma)_n(1+\alpha-2n)_n} L_n^{(\alpha-2n)}(y)\,.\,x^n, \quad \ldots \quad \ldots \quad \ldots \tag{2.2}$$

$$|x| < \tfrac{1}{4},\ \mathrm{Re}\,(\alpha) > -1.$$

$$H_6(\alpha,\ \gamma,\ -xy,\ y) = \sum_{m=0}^{\infty} \frac{1}{(\gamma)_m} \psi_m(\alpha,\ x)\,.\,y^m, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \tag{2.3}$$

$$|xy| < \tfrac{1}{4}.$$

PROOF: The L.H.S. of (2.1) is equal to

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(1+\alpha+\beta)_{2m-n}(-\beta)_n}{(\gamma)_m(m-n)!\,n!} \left(\frac{1-x}{2}\right)^{m-n} y^m(-1)^m$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(1+\alpha+\beta)_{m+n}(-\beta)_{m-n}}{(\gamma)_m\, n!\,(m-n)!} \left(\frac{1-x}{2}\right)^{n} y^m(-1)^m$$

(on reversing the inner summation)

$$= \sum_{m=0}^{\infty} \frac{(1+\alpha+\beta)_m}{(\gamma)_m} P_m^{(\beta-m,\ \alpha+m)}(x)\,.\,y^m \quad (\text{by } (1.1))$$

which proves (2.1).

Proceeding as above, summing the series for m and using (1.2), we get (2.2).

Again proceeding on similar lines, summing the series for n and using (1.3), we get (2.3).

§ 3. In this section we prove the following formulae:

$$\Phi_3(-\beta,\ \gamma,\ -x,\ -xy) = \sum_{m=0}^{\infty} \frac{1}{(\gamma)_m} L_m^{(\beta-m)}(y)\,.\,x^m, \quad \ldots \quad \ldots \tag{3.1}$$

$$\mathrm{Re}\,(\beta) > -1.$$

$$\Phi_1(\alpha,\ \beta,\ \gamma,\ -xy,\ y) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n(\beta-n)_n} \psi_n(\beta-n,\ x)\,.\,y^n, \quad \ldots \tag{3.2}$$

$$|xy| < 1.$$

PROOF OF (3.1): The L.H.S. of (3.1) is equal to

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-\beta)_{m-n}}{(\gamma)_m(m-n)!\,n!}(-1)^m x^m y^n$$

$$=\sum_{m=0}^{\infty}\frac{1}{(\gamma)_m}L_m^{(\beta-m)}(y)\,.\,x^m \text{ (by (1.2))}$$

which proves (3.1).

PROOF OF (3.2): The L.H.S. of (3.2) is equal to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_n(\beta)_m}{(\gamma)_n\, m!\,(n-m)!}(-1)^m x^m y^n$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n(\beta-n)_n}\psi_n(\beta-n,\, x)\,.\,y^n \text{ (by (1.3))}$$

which proves (3.2).

§ 4. The formulae, to be proved here, are

$$(1-x)^a {}_3\Phi_A^{(2)}(a,\, b_1;\; b_1, -c_2,\, 1+c_3;\; x,\, y(1-x),\, y(1-x)(1-z)/2)$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n}{(-c_2)_n(1+c_3)_n}P_n^{(c_3,\, c_2-c_3-2n)}(z)\,.\,y^n, \qquad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

$|x|<1$, Re $(c_3)>-1$, Re $(c_2-c_3)>-1$.

$$(1-x)^{\alpha_1}F_E(\alpha_1,\, \alpha_1,\, \alpha_1,\, \beta_1,\, \beta_2,\, \beta_2;\; \beta_1,\, -\gamma_2,\, 1+\gamma_3;\; x,\, (1-x)y,\, (1-x)(1-z)y/2)$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\beta_2)_n}{(-\gamma_2)_n(1+\gamma_3)_n}P_n^{\gamma_3,\, \gamma_2-\gamma_3-2n}(z)\,.\,y^n, \qquad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

$|x|<r$, $|(1-x)y|<s$, $|(1-x)y(1-z)/2|<t$, $r+(\sqrt{s}+\sqrt{t})^2=1$,
Re $(\gamma_3)>-1$, Re $(\gamma_2-\gamma_3)>-1$.

$$(1-x)^{-b_1}{}_3\Phi_M^{(4)}(a_1, -b_1, -b_1;\; a_1,\, c_2,\, c_2;\; x,\, -yz,\, -z(1-x))$$

$$=\sum_{p=0}^{\infty}\frac{1}{(c_2)_p}L_p^{(b_1-p)}(y)\,.\,z^p;\; |x|<1,\ \text{Re}\,(b_1)>-1. \qquad \ldots \quad \ldots \quad (4.3)$$

$$(1-x)^{b_1}{}_3\Phi_M^{(4)}(a_1,\, b_1,\, b_1;\; a_1,\, c_2,\, c_2;\; x,\, y,\, -yz(1-x))$$

$$=\sum_{n=0}^{\infty}\frac{1}{(c_2)_n(b_1-n)_n}\psi_n(b_1-n,\, z)\,.\,y^n;\; |x|<1. \qquad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

$$(1-x)^{a_1}{}_3\Phi_G^{(1)}(a_1,\, a_1,\, a_1,\, b_1,\, b_2;\; b_1,\, c_2,\, c_2;\; x,\, yz(x-1),\, z(x-1))$$

$$=\sum_{p=0}^{\infty}\frac{(a_1)_p}{(c_2)_p(b_2-p)_p}\psi_p(b_2-p,\, y)\,.\,z^p;\; |x|<1,\ |yz(x-1)|<1. \qquad (4.5)$$

$$(1-x)^{a_1}{}_3\Phi_G^{(1)}(a_1,\, a_1,\, a_1,\, b_1, -b_2;\; b_1,\, c_2,\, c_2;\; x,\, y(x-1),\, yz(x-1))$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n}{(c_2)_n}L_n^{(b_2-n)}(z)\,.\,y^n;\; |x|<1,\ |y(x-1)|<1,\ \text{Re}\,(b_2)>-1. \quad (4.6)$$

PROOF: The L.H.S. of (4.1) is equal to

$$(1-x)^{a} \sum_{n,\,p=0}^{\infty} \left[\frac{(a)_{n+p} y^{n+p} (1-x)^{n+p}}{n!\, p!\, (-c_2)_n (1+c_3)_p} \left(\frac{1-z}{2}\right)^{p} \sum_{m=0}^{\infty} \frac{(a+n+p)_m}{m!} x^m \right]$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(a)_n}{(n-p)!\, p!\, (-c_2)_{n-p} (1+c_3)_p} y^n \left(\frac{1-z}{2}\right)^{p}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(-c_2)_n (1+c_3)_n} P_n^{(c_3,\, c_2-c_3-2n)}(z) \,.\, y^n \text{ (by (1.1))}$$

which proves (4.1).

Proceeding on similar lines as above, the formulae (4.2), (4.3), (4.4), (4.5) and (4.6) can be proved easily.

§ 5. *Particular cases*—In (2.3), putting $\gamma = \alpha$, applying the first part of exercise 10 (Rainville 1960, p. 70), we get

$$(1+4xy)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1+4xy}}\right)^{\alpha-1} \exp\left(\frac{2y}{1+\sqrt{1+4xy}}\right) = \sum_{m=0}^{\infty} \frac{\psi_m(\alpha, x)}{(\alpha)_m} y^m \qquad (5.1)$$

which is due to Burchnall (1951).

In (2.3), putting $\gamma = \alpha = 1$ and proceeding as above, we get

$$(1+4xy)^{-\frac{1}{2}} \exp\left(\frac{2y}{1+\sqrt{1+4xy}}\right) = \sum_{m=0}^{\infty} \frac{\psi_m(1, x)}{m!} y^m \qquad .. \quad (5.2)$$

which is due to Krall and Frink (1949).

In (3.1), putting $\gamma = -\beta$, we obtain

$$\sum_{m=0}^{\infty} \frac{1}{m!}\, {}_0F_1\left[\begin{matrix} - \,, \\ -\beta+m, \end{matrix}\; xy\right] x^m = \sum_{m=0}^{\infty} \frac{(\beta-m)!}{(\beta)!} L_m^{(\beta-m)}(y) \,.\, x^m. \qquad .. \quad (5.3)$$

In (3.2), putting $\alpha = \gamma$, we have

$$e^{y}(1+xy)^{-\beta} = \sum_{n=0}^{\infty} \frac{1}{(\beta-n)_n} \psi_n(\beta-n, x) \,.\, y^n. \qquad .. \quad .. \quad (5.4)$$

In (4.1), taking $c_2 = -a$, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, {}_1F_1\left[\begin{matrix} a+n\,; \\ 1+c_3\,; \end{matrix}\; \frac{y(1-z)}{2}\right] y^n = \sum_{n=0}^{\infty} \frac{1}{(1+c_3)_n} P_n^{(c_3,\, -a-c_3-2n)}(z) \,.\, y^n. \qquad (5.5)$$

In (4.4), putting $c_2 = b_1$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, {}_0F_1\left[\begin{matrix} -\,; \\ b_1+n\,; \end{matrix}\; y\right] y^n z^n = \sum_{n=0}^{\infty} \frac{1}{(b_1)_n (b_1-n)_n} \psi_n(b_1-n, z) \,.\, y^n. \qquad (5.6)$$

Letting $c_2 = a_1$, the generating relation (4.6) degenerates to the well-known result (Erdélyi 1953, vol. II, p. 189 (19))

$$e^{-yz}(1+y)^{b_2} = \sum_{n=0}^{\infty} L_n^{(b_2-n)}(z) \, . \, y^n. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5.7)$$

Moreover, replacing α by $a-1$ and x by $-x/b$ in (2.3), and using (1.4), we get the generating relation for Krall-Frink generalized Bessel polynomials

$$H_6\left(a-1, \gamma, \frac{xy}{b}, y\right) = \sum_{m=0}^{\infty} \frac{(a-1)_m}{(\gamma)_m m!} y_m(a, b, x) \, . \, y^m, \quad \left|\frac{xy}{b}\right| < \tfrac{1}{4}. \quad \ldots \quad (5.8)$$

In the same manner, we can easily transform the results (3.2), (4.4) and (4.5) into the form of the Krall-Frink generalized Bessel polynomials.

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