STRUCTURE OF A CYLINDRICAL POLYTROPE

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Structure of an infinite self-gravitating cylinder in which pressure and density is related by $P =$ constant $\rho^{1+\frac{1}{n}}$ has been studied. Physical and mathematical conditions in the immediate neighbourhood of the origin along the radius have also been studied. Solutions for n tending to 0 and -1 have been discussed.

1. INTRODUCTION

The equilibrium theory of a self-gravitating cylinder has quite recently been investigated (Ostriker 1964). Ostriker (1964) and Randers (1942) have already discussed the physical validity of such mathematical problems. In this paper the author has shown that in the immediate neighbourhood of the origin, physical and mathematical conditions do not change as we change the index of the polytrope. The magnitude of this immediate neighbourhood has been discussed. As we leave this immediate neighbourhood of the origin, there is an interfacial region to which solutions, relevant at the origin and at other points on the radius, are relevant simultaneously. Solutions for $n = 0$ and n tending to 0 have been differentiated.

2. FUNDAMENTAL EQUATIONS OF THE PROBLEM

If P and ρ be the pressure and density within the cylinder, the equation of hydrostatic equilibrium is

$$\Delta\phi = \frac{1}{\rho}\,\Delta P. \qquad \ldots \quad (1)$$

The second equation of equilibrium is given by Poisson's equation which, for an infinite long cylinder, may be written as

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r\,d\phi}{dr}\right) = -4\pi G\rho. \qquad \ldots \quad (2)$$

The fundamental equations of our problem are (1), (2) and

$$P = K_n\rho^{1+\frac{1}{n}}. \qquad \ldots \quad (3)$$

3. Structure of an Infinite Self-gravitating Polytropic Cylinder

(a) *Mathematical Conditions at the Origin*

The appropriate form of the Lane-Emden equation due to Ostriker is

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\theta}{d\xi}\right)=\theta''+\frac{1}{\xi}\theta'=-\theta^n. \qquad (4)$$

The valid solutions satisfy the boundary conditions

$$\theta(0)=1\,;\ \theta'(0)=0. \qquad (5)$$

It is clear from eqns. (3) and (5) that at the origin pressure, density and temperature gradient do not change as we change the index of the configuration. Hence the behaviour of solutions in the immediate neighbourhood of the origin should be the same in all the polytropes.

The vanishing of pressure and density at the origin shows that at the origin pressure and density are not functions of the radius vector. Fundamental equations suggest that any point in the configuration at which relation (3) is valid, both P and ρ are functions of r. Hence at the origin, our pressure-density relation can be relevant only for n tending to 0 or -1. The reason being that for these limiting values of n pressure and density tend to become independent of each other.

Now we have to differentiate the physical circumstances in which the solutions for these two values of n will be relevant. Let us assume a region in which at every point pressure remains finite but pressure gradient vanishes. Then a pressure-density relation of the kind given in (3), valid in such a region, will be valid at the origin also for the reason that every point of the region as well as the origin is governed by the same physical and mathematical conditions. Naturally in the assumed region eqn. (3) can be relevant only for n tending to -1. If we assume a region in which at every point density is finite and density gradient vanishes, then similar arguments show that at the origin relation (3) is valid for n tending to 0. We therefore conclude that if we use $(r;\,P)$-variables in the study of mathematical structure of cylindrical polytropes, then solutions for n tending to -1 give the arrangement of solutions in the immediate neighbourhood of the origin and if we use $(r;\,\rho)$-variables, the arrangement of solutions is given by n tending to 0.

(b) *Solutions for n tending to* 0 *and* -1

Now we shall see that when the fundamental equation in terms of r and P is reduced to that of a first order in $(u_P;\,v_P)$-plane, we get finite solutions for n tending to 0 and -1. Solutions for these two values of n are, however, not obtainable in $(u_\rho;\,v_\rho)$-variables. We thus see that natural variables r and P are most suitable to use in the study of the structure of a cylindrical polytrope.

Cylindrical polytropic equation in $(u_P;\ v_P)$*-variables*—In terms of ξ_P and P, the Lane-Emden equation for polytropic cylinder can be expressed as

$$\frac{d^2P}{d\xi_P{}^2}-\frac{n}{n+1}\,\frac{1}{P}\left(\frac{dP}{d\xi_P}\right)^2+\frac{1}{\xi}\,\frac{dP}{d\xi_P}=-P^{\frac{2n}{n+1}} \qquad (6)$$

where

$$r=\alpha_P\xi_P\,;\quad \alpha_P=\left[\frac{K_n^{\frac{2n}{n+1}}}{4\pi G}\right]^{\frac{1}{2}}. \qquad (7)$$

In the usual manner eqn. (6) can be reduced to that of the first order:

$$\frac{u_P}{v_P}\,\frac{dv_P}{du_P}=-\frac{(n+1)u_P+v_P}{(n+1)u_P+nv_P-2(n+1)} \qquad (8)$$

where

$$u_P=-\frac{\xi_P P^{\frac{2n}{n+1}}}{P'}\,;\quad v_P=-\frac{\xi_P P'}{P}. \qquad (9)$$

(a) $n=0$.—For $n=0$, eqn. (8) becomes

$$\frac{1}{v_P{}^2}\,\frac{dv_P}{du_P}+\frac{1}{u_P-2}=-\frac{1}{u_P(u_P-2)}. \qquad (10)$$

The foregoing is reducible to a linear form. The primitive is

$$\frac{1}{v_P}\,e^{-\int\frac{du_P}{u_P-2}}=\int\frac{1}{u_P(u_P-2)}\,e^{-\int\frac{du_P}{u_P-2}}+\log c_1 \qquad (11)$$

where c_1 is the constant of integration. Simplifying and integrating the above equation, we get

$$\frac{1}{v_P}=(u_P-2)\left[\log\left(\frac{u_P}{u_P-2}\right)^{\frac{1}{2}}c_1-\tfrac{1}{2}(u^P-2)\right] \qquad (12)$$

For $n=0$, $\theta=P$. Hence for $n=0$, equation in $(u_\theta;\ v_\theta)$ and $(u_P;\ v_P)$-variables will be identical. It is interesting to note at this stage that whether we first reduce eqn. (6) to (8) and then put $n=0$ or we reduce

$$\frac{d^2P}{d\xi_P{}^2}+\frac{1}{\xi_P}\,\frac{dP}{d\xi}=-1 \qquad (13)$$

by

$$u_P=-\frac{\xi_P}{P'}\,;\ v_P=-\frac{\xi\theta'}{P} \qquad (14)$$

in both the cases we reach to eqn. (10); nevertheless the solutions of (13) and (12) are different. The reason is that the known solution of (13) corresponds to $n=0$ in our pressure-density relation and solution (12) corresponds to n tending to 0. This may be seen as follows: The solution of (13) naturally corresponds to $n=0$ but if we first integrate (4) and then put $n=0$ in the first integral, then the solution will correspond to n tending to 0. In other words, as long as the first integration is not carried out, P' in (6) as well in (13) corresponds to the same solution, i.e. to n tending to 0. In the detailed process of reduction to $(u_P;\ v_P)$-variables, we substitute P'' in terms of P', P and ξ_P. Whether we substitute P'' from eqn. (6) without putting $n=0$ or from eqn. (13), P' always corresponds to n tending to 0. This is the reason that the solution (10) controls the equilibrium of an infinite self-gravitating cylinder in which a polytropic relation of the kind

$$\rho = P^{\epsilon} \qquad \ldots \quad (15)$$

where ϵ is an arbitrarily small quantity, subsists and known solution for $n=0$ controls the equilibrium of a liquid cylinder.

(b) $n=-1$.—For $n=-1$, eqn. (8) gives

$$\frac{u_P}{v_P}\frac{dv_P}{du_P}=1. \qquad \ldots \quad (16)$$

The integration of the above equation gives

$$v_P = c'u_P \qquad \ldots \quad (17)$$

which is a straight line passing through the origin at an angle $\tan^{-1} c'$ where c' is the constant of integration.

Solutions in $(u_\rho;\ v_\rho)$-variables.—In terms of ξ_ρ and ρ our Lane-Emden equation for cylindrical polytrope is

$$\frac{d^2\rho}{d\xi_\rho{}^2}+\frac{1-n}{n}\frac{1}{\rho}\frac{d\rho}{d\xi_\rho}+\frac{1}{\xi_\rho}\frac{d\rho}{d\xi_\rho}=-\rho^{2-\frac{1}{n}} \qquad \ldots \quad (18)$$

where

$$r=\alpha_\rho\xi_\rho\,;\ \alpha_\rho=\sqrt{\frac{k(n+1)}{4\pi\, nG}}. \qquad \ldots \quad (19)$$

The substitutions

$$u_\rho=-\frac{\xi_\rho\rho^{2-\frac{1}{n}}}{\rho'}\,;\ v_\rho=-\frac{\xi\rho'}{\rho} \qquad \ldots \quad (20)$$

transform (18) into

$$\frac{u_\rho}{v_\rho}\frac{dv_\rho}{du_\rho}=-\frac{nu_\rho+(1-n)v_\rho}{n(u_\rho+v_\rho-2)}. \qquad \ldots \quad (21)$$

It appears that neither for $n=0$ nor for $n=-1$, eqn. (21) gives a finite solution.

Since $(u_P;\ v_P)$ and $(u_\rho;\ v_\rho)$-variables are related, one may expect that solutions (12) and (17) may possibly be expressed in $(u_\rho;\ v_\rho)$-variables. We shall see below that even this is not possible.

From our pressure-density relation we easily obtain

$$P^{\frac{2n}{n+1}}=K_n^{\frac{2n}{n+1}}\rho^2\,;\ P'=K_n\left(1+\frac{1}{n}\right)\rho^{1/n}\rho'. \qquad \ldots \quad (22)$$

Further from (7) and (19), we get

$$\frac{\xi_P}{\xi_\rho} = \left(\frac{n+1}{n} K_n^{\frac{1-n}{1+n}}\right)^{\frac{1}{2}}. \quad \ldots \quad (23)$$

With the help of (22) and (23), we can rewrite eqn. (9) as

$$u_P = -\left(\frac{n}{n+1} K_n^{\frac{n-1}{n+1}}\right)^{\frac{1}{2}} u_\rho. \quad \ldots \quad (24)$$

Equation (24) shows that neither for $n = 0$ nor for $n = -1$ we can express u_P in terms of u_ρ. Further for $n = -1$, α_P is either 0 or infinite depending on whether K_{-1} is less than one or greater than one. For $K_{-1} = 1$, α_P is indeterminate. This suggests that the natural variables r and P are most suitable to use in the study of the structure of cylindrical polytropes.

(c) *The Immediate Neighbourhood of the Origin*

We have seen above that solution (17) is relevant in the immediate neighbourhood of the origin. Now we shall develop an expression giving the range of the immediate neighbourhood.

Let us now assume a series expansion at the origin of the form

$$P = 1+\alpha\xi_P^2+\beta\xi_P^4+\ldots \quad \ldots \quad (25)$$

The above series expansion shall satisfy the boundary conditions

$$P(0) = 1\,;\; P'(0) = 0. \quad \ldots \quad (26)$$

Substituting the series (25) in (6) and equating the coefficient of the like powers, we find that the first three terms of the series are

$$P = 1+\tfrac{1}{4}\xi_P^2+\frac{3n}{64(n+1)}\xi_P^4+\ldots \quad \ldots \quad (27)$$

Taking only the first three terms of the above series, after a few simplifications u_P and v_P may be expressed as

$$\left.\begin{aligned} u_P &= -\frac{1}{2(n+1)}\,\frac{32(n+1)^2-16n(n+1)\xi_P^2+3n^2\xi_P^4}{3n\xi_P^2-8(n+1)} \\ v_P &= -4\,\frac{3n\xi_P^4-8(n+1)\xi_P^2}{64(n+1)-16(n+1)\xi_P^2+3n\xi_P^4} \end{aligned}\right\}. \quad \ldots \quad (28)$$

A homologous family in (ξ_P; P)-plane yields only one curve in (u_P; v_P)-plane. Let $u(n;\, \xi_P)$, $v(n;\, \xi_P) \equiv E(n)$-curve. The point of intersection of $E(n)$-curve and solution (17) is given by

$$4\,\frac{3n\xi_P^4-8(n+1)\xi_P^2}{64(n+1)-16(n+1)\xi_P^2-3n\xi_P} = \frac{c'}{2(n+1)}\,\frac{32(n+1)^2-16n(n+1)\xi_P^2+3n^2\xi_P^4}{3n\xi_P^2-8(n+1)}. \quad \ldots \quad (29)$$

Since we are considering the immediate neighbourhood of the origin, we may retain terms in ξ_P^2 only and neglect the higher powers of ξ_P. After minor simplifications, eqn. (29) yields

$$\xi_P^2 = \frac{4c'(n+1)}{(n+1)+(3n+1)c'}. \qquad \ldots \quad \ldots \quad \ldots \quad (30)$$

The above expression gives the range of ξ_P in which solution (17) is valid. The boundary conditions of the problem suggest that c' is an arbitrary small quantity. The vanishing of ξ_P for $n = -1$ supports our investigation that at the origin there is a restricted immediate neighbourhood to which solutions for n tending to -1 are relevant.

(d) *The Interfacial Region*

As we leave the immediate neighbourhood of the origin given by (30), relation (3) can be relevant at any point on the radius; but since there must be a continuity, there must subsist an interfacial region in which solutions valid in the immediate neighbourhood of the origin as well as valid at other points on the radius are valid simultaneously.

Mass of the cylinder per unit length is

$$M(\xi_P) = 2\pi \int_0^R r\,\rho(r)\,dr = 2\pi\,\alpha_P^2 \left[\xi_P P^{-\frac{n}{n+1}} \frac{dP}{d\xi_P}\right]_0^R. \qquad \ldots \quad \ldots \quad (31)$$

The radius of the cylinder is

$$R = \left[\frac{K^{\frac{2n}{n+1}}}{4\pi G}\right]^{\frac{1}{2}} \xi_{P_1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where P has its first 0 at $\xi_P = \xi_{P_1}$. Internal energy of the cylinder per unit length is

$$U = \frac{1}{\gamma-1}\,\alpha_P \int P\,d\xi_P. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

Let $P = K_n \rho^{1+\frac{1}{n}}$ be relevant for n_0 and n at the origin and at other points on the radius. The value of K_n, however, is the same in the entire configuration. Let $(\xi_{P_0};\, P_0)$ and $(\xi_P;\, P)$ be the variables used at the origin and at the other points on the radius. At the point of interface, the values of ρ, M, R and U for the two mentioned sets of variables should be identical. Thus our equations of fit are

$$\left.\begin{aligned} P_0^{\frac{n_0}{n_0+1}} &\equiv P^{\frac{n}{n+1}} \\ \xi_{P_0} P_0^{-\frac{n}{n+1}} \frac{dP_0}{d\xi_{P_0}} &\equiv \xi_P P^{-\frac{n}{n+1}} \frac{dP}{d\xi_P} \\ \xi_{P_0} &\equiv \xi_P \\ P_0 &\equiv P \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (34)$$

The above equations of fit may easily be expressed as

$$u_P(n_0\,;\,\xi_{P_0}) \equiv u_P(n\,;\,\xi_P)\,;\; v_P(n_0\,;\,\xi_{P_0}) \equiv v_P(n\,;\,\xi_P). \qquad (35)$$

As we leave this interfacial region along the radius, physical and mathematical conditions are given by solutions of (6) for the corresponding value of n.

(e) *Physical Conditions in the Immediate Neighbourhood of the Origin*

So far we have discussed the values of n for which our pressure-density relation is valid at different points on the radius of the cylinder. Now we shall discuss the physics of the immediate neighbourhood of the origin in greater detail.

In the immediate neighbourhood of the origin, pressure-density relation is valid for n tending to 0 and -1 only. For these two limiting values of n, mathematically P and ρ are related but not physically. That is, for these values of n, if the pressure distribution is known, we cannot directly derive from there the density distribution and vice versa. This is further clear from the fact that solutions for n tending to 0 and -1 in $(u_P;\ v_P)$-variables cannot be expressed in $(u_\rho;\ v_\rho)$-variables. Thus we must study pressure and density distribution in the immediate neighbourhood of the origin independently.

We have already seen that in the immediate neighbourhood of the origin solutions for n tending to -1 will be relevant if we use r and P variables in the study of the structure of the polytropes and solutions for n tending to 0 will be relevant if we use r and ρ variables. The physical interpretation of this mathematical conclusion is that pressure distribution in the immediate neighbourhood of the origin is given by solution for n tending to -1 and the density distribution is given by solutions for n tending to 0. That is, eqn. (17) gives the pressure distribution and eqn. (21) for $n \to 0$ gives the density distribution. The immediate neighbourhood of the origin behaves as a polytropic cylinder in which both pressure and density tend toward constancy.

Eddington used the results of a sphere of uniform density to the minimal problems of stars. In a similar way the results of a polytropic cylinder of uniform density might prove useful in the study of the minimal problems of a polytropic cylinder.

4. Conclusions

1. Mathematical and physical conditions in the immediate neighbourhood of the origin do not change as we change the index of the polytrope.

2. In the immediate neighbourhood of the origin, if we know the pressure distribution, we cannot directly derive from there density distribution and vice versa. The pressure distribution in this neighbourhood is given by solution (17) and the density distribution is given by eqn. (21) for $n = 0$.

3. Our natural variables r and P are, however, most suitable to use in the study of the structure of a cylindrical polytrope. In the scale of ξ_P where ξ_P is linearly related with r, the range in which the solution (17) is valid at the centre is given by (30).

4. As we leave the immediate neighbourhood of the origin along the radius, given by eqn. (30), there is an interfacial region in which two sets of equations, one valid at the origin and the other valid at other points on the radius, are valid simultaneously. Our equations of fit are given by (35).

5. Solutions for n tending to 0 and -1 are given by eqns. (12) and (17) respectively.

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