ON THE EXISTENCE OF PERIODIC SOLUTIONS FOR A SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS

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A class of non-linear differential equations of the type $\frac{dx}{dt} = f(t, x) + g(t, x)$ is studied for establishing the existence of a periodic solution under suitable conditions. The notion of asymptotic equivalence together with the Brouwer's theorem is used to establish the existence of periodic solutions. The result generalizes an earlier result of Perov as given by Krasnoselskii (1968).

§ 1. To establish existence of periodic solutions of non-linear systems is normally a difficult problem. Some interesting theorems concerning this are available in a recent monograph of Krasnoselskii (1968). We study a class of non-linear differential equations and obtain conditions for the existence of a periodic solution. This generalizes a theorem of Perov as given by Krasnoselskii (1968). We shall make use of the following fixed point theorem of Brouwer in our main theorem.

Theorem of Brouwer

If $S \subset R^n$, is closed, bounded and convex and T is a continuous map from S to S, then T has at least one fixed point.

We consider the following two systems of differential equations

$$\frac{dx}{dt} = f(t, x) \qquad \ldots \quad (1)$$

$$\frac{dy}{dt} = f(t, y) + g(t, y) \qquad \ldots \quad (2)$$

where x and y are n-vectors and f and g are n-vector functions satisfying the following properties:

(i) $f(t, x)$ and $g(t, x)$ are continuous for $t \geq t_0$ and $x \in R^n$.

(ii) $f(t, x)$ and $g(t, x)$ are periodic of period ω for each fixed x.

(iii) $| \langle y-x, f(t, y)-f(t, x) \rangle | \leq L \| x-y \|^2$ where '$\langle\ \rangle$' denotes scalar multiplication.

The above conditions assert the local existence of solutions of (1) and (2) and uniqueness of (1). Further suppose (iv) $f(t, x)$ and $g(t, x)$ are such that there exist unique solution for (1) and (2) in the interval $(t_0, t_0+\omega)$ with any initial condition $x(t_0) = x_0 \in R^n$ and $y(t_0) = y_0 \in R^n$. Explicit conditions can be easily stated.

§ 2. With the above preliminary discussion, we go over to the main aspect of this note. We require the following lemma.

Lemma—If the systems of differential equations (1) and (2) satisfy the conditions stated in § 1, along with the following two conditions

(*a*) $\dfrac{\| g(t, y) \|}{\| y \|} \to 0$ as $\| y \| \to \infty$ where '$\| \quad \|$' denotes Euclidean norm in R^n,

(*b*) $\| x(t, t_0, z_0) \| \leq [k \| z_0 \|^p + P]$ for $0 \leq p \leq 1$ in $t \in [t_0, t_0+\omega]$, and P is a constant,

then

$$\frac{\| y(t, t_0, z_0) - x(t, t_0, z_0) \|}{\| z_0 \|^p} \to 0 \text{ as } \| z_0 \| \to \infty.$$

PROOF:

$$\text{Let } r = \sup_{t \in [t_0, t_0+\omega]} \| y(t) \| \text{ and } \lambda = \sup_{t \in [t_0, t_0+\omega]} \| y(t) - x(t) \|$$

consider $<y-x, y'-x'>$ where $y' = \dfrac{dy}{dt}$, $x' = \dfrac{dx}{dt}$, $x = x(t, t_0, z_0)$ and $y = y(t, t_0, z_0)$.

Putting the values of x' and y', we have

$$<y-x, y'-x'> = <y-x, f(t, y)-f(t, x)> + <y-x, g(t, y)>$$

or

$$\int_{t_0}^{t} d[\| y(s) - x(s) \|] = \int_{t_0}^{t} 2<y-x, f(s, y)-f(s, x)> ds$$

$$+2 \int_{t_0}^{t} <y-x, g(s, y)> ds$$

or

$$\| y(t) - x(t) \|^2 \leq \int_{t_0}^{t} 2L \| y(s) - x(s) \|^2 ds + 2 \int_{t_0}^{t} \| y-x \| \, \| g(t, y) \| ds$$

or

$$\| y(t) - x(t) \|^2 \leq \int_{t_0}^{t} 2L \| y(s) - x(s) \|^2 ds + 2\lambda \int_{t_0}^{t} [M + \epsilon \| y(s) \|] \, ds$$

since

$$\frac{\| g(t, y) \|}{\| y \|} \to 0 \text{ we have for any } \epsilon > 0, \exists M(\epsilon), \ni \| g(t, y) \| \leq M + \epsilon \| y \|.$$

Thus we have

$$\| y(t) - x(t) \|^2 \leq \int_{t_0}^{t} 2L \| y(s) - x(s) \|^2 ds + 2\lambda M \omega + 2\lambda \epsilon \omega r.$$

Now applying Gronwall's inequality, we have

$$\| y(t) - x(t) \|^2 \leq (2M\lambda\omega + 2\lambda \epsilon \omega r) e^{2L(t-t_0)}.$$

So

$$\lambda^2 \leq (2M \lambda\omega + 2\lambda \epsilon\omega r) \, e^{2L\omega}$$

or

$$\lambda \leq 2M\,\omega + 2\epsilon\omega\, r\, e^{2L\omega}. \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

By using condition (*b*) of the lemma, we get

$$\| y(t) \| \leq \| x(t) \| + \| x(t) - y(t) \| \leq k \| z_0 \|^p + P + 2M\,\omega\, e^{2L\omega} + 2\epsilon\omega\, r\, e^{2L\omega}$$

or

$$r \leq k \| z_0 \|^p + P + 2M\,\omega\, e^{2L\omega} + 2\epsilon\omega\, r\, e^{2L\omega}.$$

Now taking ϵ sufficiently small, we have

$$r \leq \frac{\| z_0 \|^p + P + 2M\,\omega\, e^{2L\omega}}{1 - 2\omega\epsilon\, e^{2L\omega}}.$$

So from (3) we obtain

$$\lambda \leq 2M\,\omega + 2\omega\, e^{2L\omega}\,\epsilon \left[\frac{k \| z_0 \|^p + P + 2M\,\omega\, e^{2L\omega}}{1 - 2\omega\, e^{2L\omega}\,\epsilon} \right].$$

Dividing by $\| z_0 \|^p$, we obtain

$$\overline{\lim_{\| z_0 \| \to \infty}} \frac{\lambda}{\| z_0 \|^p} \leq \frac{\epsilon k\omega\, e^{2L\omega}}{1 - 2\omega\, e^{2L\omega}\,\epsilon}.$$

Since ϵ is arbitrary, we obtain the required result.

Remark: If $\frac{dx}{dt} = f(t, x)$ where $f(t, cx) = c^k f(t, x)$ and the zero solution is uniformly asymptotically stable then Hahn (1967) points out that

$$\| x(t, t_0, x_0) \| \leq \left(a \| x_0 \|^{\frac{1}{1-k}} + c_1(t - t_0) \right)^{\frac{1}{1-k}} \quad \text{for } k > 1$$

and

$$\| x(t, t_0, x_0) \| \leq a \| x_0 \| e^{-b(t-t_0)} \qquad \text{for } k = 1.$$

Obviously homogeneity of degree one does not imply linearily and hence the condition (*b*) of the lemma is not an artificial condition for non-linear cases.

We now proceed to prove our main theorem which asserts the existence of a periodic solution for (2). We recollect that the necessary and sufficient condition for the existence of periodic solution of (2) is the existence of a fixed point for the mapping $y(t_0+\omega, t_0, z)$ mapping $z \in R^n$ to R^n. We make use of this below.

Let U and V denote translation operators defined by $Uz = y(t_0+\omega, t_0, z)$ and $Vz = x(t_0+\omega, t_0, z)$. Obviously because of continuity of $f(t, x)$ and $g(t, x)$ and uniqueness of solutions of (1) and (2), it is clear that U and V are continuous operators (Coddington and Levinson 1955). Now we prove the following theorem.

Theorem—Let $f(t, x)$ and $g(t, x)$ satisfy all the conditions of the lemma and let V not have 1 as one of its eigenvalues, then eqn. (2) has a periodic solution.

PROOF: Consider the operator $Tz = [I - V]^{-1}[Uz - Vz]$. It is easy to verify that the fixed points of T are the fixed points of U and conversely. Thus in order to show the existence of a fixed point of U, it is sufficient to show the existence of a fixed point of T.

Now

$$\| Tz \| \leq \| I - V^{-1} \| \quad \| Uz - Vz \| \leq k \| Uz - Vz \|.$$

Dividing both sides by $\| z_0 \|^p$, we have

$$\frac{\| Tz \|}{\| z \|^p} \leq k \frac{\| U(z) - V(z) \|}{\| z \|^p}.$$

By the above lemma we have

$$\lim_{\| z \|^p \to \infty} \frac{\| Uz - Vz \|}{\| z \|^p} = 0.$$

So

$$\frac{\| Tz \|}{\| z \|^p} \to 0 \text{ as } \| z \| \to \infty,$$

hence

$$\frac{\| Tz \|}{\| z \|} \to 0 \text{ as } \| z \| \to \infty \text{ as } 0 < p \leq 1. \quad \text{So } \| Tz \| \leq \bar{M}(\epsilon) + \epsilon \| z \|.$$

Now choosing $\epsilon = \frac{1}{4}$, we denote $(\bar{M})\frac{1}{4} = M_0$.
Let $S = \{ z \colon \| z \| \leq 2M_0 \}$, then $\| Tz \| \leq M_0 + \frac{1}{4} \| z \| \leq M_0 + 2\frac{1}{4} M_0$ for $z \in S$.
Thus, we have

$$\| Tz \| < 2M_0 \text{ and hence } T \text{ is a continuous mapping from } S \text{ to } S.$$

Since S is closed convex and bounded in R^n, applying Brouwer's theorem we get the required result.

It is easily verified that theorem 3.4, chap. I of Krasnoselskii (1968) is a particular case of the above theorem.

References

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