A SPATIALLY FLAT SPACE-TIME IN EINSTEIN'S UNIFIED FIELD THEORY

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A solution of the field equations in Einstein's unified field theory has been obtained. The non-static field which describes a spatially flat space-time and is of electromagnetic origin turns out to be static. It is singular all along the z-axis.

1. Introduction

On the basis of a number of observed gravitational effects it is not unreasonable to suggest that gravitation should be electromagnetic in origin. From this point of view the absence of electromagnetic field implies a flat space-time. We desire to introduce a simple electromagnetic field to study a departure from flat space-time in the context of Einstein's unified field theory. Herein we take the total field as characterized by

$$g_{11} = \frac{g_{22}}{\rho^2} = g_{33} = -1, \quad g_{44} = C, \quad g_{\underset{\vee}{14}} = \eta \qquad \ldots \quad \ldots \quad (1.1)$$

the other components of $g_{\lambda\mu}$ being zero. C and η both are functions of ρ and t. The field equations used by us in our investigation are:

$$g_{\lambda\mu,\,\omega} - g_{\alpha\mu}\Gamma^{\alpha}_{\lambda\omega} - g_{\lambda\alpha}\Gamma^{\alpha}_{\omega\mu} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (1.2a)$$

$$\Gamma^{\alpha}_{\underset{\vee}{\lambda\alpha}} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2b)$$

$$R_{\underline{\lambda\mu}} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2c)$$

$$R_{\underset{\vee}{\lambda\mu},\,\omega} + R_{\underset{\vee}{\mu\omega},\,\lambda} + R_{\underset{\vee}{\omega\lambda},\,\mu} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (1.2d)$$

with

$$R_{\underline{\lambda\mu}} = \Gamma^{\alpha}_{\underline{\lambda\mu},\,\alpha} - \tfrac{1}{2}\left(\Gamma^{\alpha}_{\underline{\lambda\alpha},\,\mu} + \Gamma^{\alpha}_{\underline{\mu\alpha},\,\lambda}\right) - \Gamma^{\alpha}_{\underline{\lambda\beta}}\Gamma^{\beta}_{\underline{\alpha\mu}} + \Gamma^{\alpha}_{\underline{\lambda\mu}}\Gamma^{\beta}_{\underline{\alpha\beta}} - \Gamma^{\alpha}_{\underset{\vee}{\lambda\beta}}\Gamma^{\beta}_{\underset{\vee}{\alpha\mu}}$$

and

$$R_{\underset{\vee}{\lambda\mu}} = \Gamma^{\alpha}_{\underset{\vee}{\lambda\mu},\,\alpha} - \Gamma^{\alpha}_{\underline{\lambda\beta}}\Gamma^{\beta}_{\underset{\vee}{\alpha\mu}} - \Gamma^{\alpha}_{\underset{\vee}{\lambda\beta}}\Gamma^{\beta}_{\underline{\alpha\mu}} + \Gamma^{\alpha}_{\underset{\vee}{\lambda\mu}}\Gamma^{\beta}_{\underline{\alpha\beta}}. \qquad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

Here and in what follows a comma (,) and a semicolon(;) preceding a suffix denote partial differentiation and covariant differentiation with respect to the affine connections $\Gamma^{\alpha}_{\lambda\mu}$.

It is found that C and η both are singular all along the z-axis. It is further to be noted that the electromagnetic field is free of charge and current, though η is non-static. It is due to the fact that the non-static solutions are reducible to the static solutions by coordinate transformation.

2. Preliminary Calculations

The contravariant tensor $g^{\lambda\mu}$ is defined by

$$g_{\lambda\mu}g^{\lambda\omega} = g_{\mu\lambda}g^{\omega\lambda} = \delta^{\omega}_{\mu}. \qquad \qquad (2.1)$$

The non-vanishing components of $g^{\lambda\mu}$ are

$$\left.\begin{aligned} & g^{11} = -\frac{C}{C-\eta^2}, \quad g^{\underset{\vee}{14}} = -\frac{\eta}{C-\eta^2}, \\ & g^{22} = -\frac{1}{\rho^2}, \quad g^{33} = -1, \quad g^{44} = \frac{1}{C-\eta^2} \end{aligned}\right\}. \qquad \qquad (2.2)$$

The determinant g of $g_{\lambda\mu}$ is given by

$$g = -\rho^2(C-\eta^2). \qquad \qquad (2.3)$$

The non-vanishing components of $\Gamma^{\alpha}_{\lambda\mu}$ as determined from (1.2a) are:

$$\left.\begin{aligned} & \Gamma^1_{\underset{\vee}{14}} = -\frac{C}{C-\eta^2}\left(\eta_{,1}-\frac{\eta C_{,1}}{2C}\right), \quad \Gamma^4_{\underset{\vee}{14}} = \frac{1}{C-\eta^2}\left(\eta_{,4}-\frac{\eta C_{,4}}{2C}\right) \\ & \Gamma^2_{\underset{\vee}{24}} = -\frac{\eta}{\rho}, \quad \Gamma^1_{\underline{14}} = -\eta\Gamma^4_{\underset{\vee}{14}} \\ & \Gamma^1_{22} = -\rho, \quad \Gamma^1_{44} = \frac{C_{,1}}{2}+2\eta\Gamma^1_{\underset{\vee}{14}} \\ & \Gamma^2_{\underline{12}} = \frac{1}{\rho}, \quad \Gamma^4_{11} = -\frac{2\eta}{C}\Gamma^4_{\underset{\vee}{14}} \\ & \Gamma^4_{\underline{14}} = \frac{C_{,1}}{2C}+\frac{\eta}{C}\Gamma^1_{\underset{\vee}{14}}, \quad \Gamma^4_{44} = \frac{C_{,4}}{2C} \end{aligned}\right\}. \qquad \qquad (2.4)$$

From (1.3) and (2.4) we obtain the non-vanishing components of $R_{\underline{\lambda\mu}}$ and $R_{\underset{\vee}{\lambda\mu}}$ as follows:

$$\left.\begin{aligned} R_{11} = & -\frac{C_{,11}}{2C}+\left(\frac{C_{,1}}{2C}\right)^2-\frac{\eta}{C}\left(\Gamma^1_{\underset{\vee}{14},1}+2\Gamma^4_{\underset{\vee}{14},4}\right)-\frac{1}{C}\left(\eta_{,1}+\frac{\eta^2}{C}\Gamma^1_{\underset{\vee}{14}}\right)\Gamma^1_{\underset{\vee}{14}} \\ & -\frac{1}{C}\left[2\eta_{,4}-\frac{\eta C_{,4}}{C}-(C-2\eta^2)\Gamma^4_{\underset{\vee}{14}}\right]\Gamma^4_{\underset{\vee}{14}} \\ R_{\underline{14}} = & \frac{\eta}{2C}\left(\Gamma^1_{\underset{\vee}{14},4}-C\Gamma^4_{\underset{\vee}{14},1}\right)-\tfrac{1}{2}\left(\eta_{,1}-\frac{\eta C_{,1}}{C}\right)\Gamma^4_{\underset{\vee}{14}}+\frac{1}{2C}\left(\eta_{,4}-\frac{\eta C_{,4}}{C}\right)\Gamma^1_{\underset{\vee}{14}}-\left(2-\frac{3\eta^2}{C}\right)\Gamma^1_{\underset{\vee}{14}}\Gamma^4_{\underset{\vee}{14}} \\ \frac{R_{22}}{\rho} = & -\frac{1}{2C}\left(C_{,1}+2\eta\Gamma^1_{\underset{\vee}{14}}\right) \\ R_{44} = & \tfrac{1}{2}\left(C_{,11}+\frac{C_{,1}}{\rho}-\frac{C^2_{,1}}{2C}\right)+\eta\left(2\Gamma^1_{\underset{\vee}{14},1}+\Gamma^4_{\underset{\vee}{14},4}\right)+\left[2\eta_{,1}+\eta\left(\frac{2}{\rho}-\frac{3C_{,1}}{2C}\right)+\left(1-\frac{2\eta^2}{C}\right)\Gamma^1_{\underset{\vee}{14}}\right]\Gamma^1_{\underset{\vee}{14}} \\ & +\left(\eta_{,4}-\frac{\eta C_{,4}}{2C}-\eta^2\Gamma^4_{\underset{\vee}{14}}\right)\Gamma^4_{\underset{\vee}{14}}+\frac{\eta^2}{\rho^2} \\ R_{\underset{\vee}{14}} = & \Gamma^1_{\underset{\vee}{14},1}+\Gamma^4_{\underset{\vee}{14},4}+\frac{1}{\rho}\Gamma^1_{\underset{\vee}{14}}+\frac{\eta}{\rho^2} \end{aligned}\right\}. \qquad \qquad (2.5)$$

3. The Field Equations and Their Solutions

The form assumed by the field eqns. (1.2*b*–1.2*d*), when the various simplifications have been made, may be expressed as follows:

$$\frac{2\eta_{,4}}{\eta}-\frac{C_{,4}}{C}=0 \quad \ldots \quad (3.1a)$$

$$\frac{C}{C-\eta^2}\left(\eta_{,1}-\frac{\eta C_{,1}}{2C}\right)+\frac{\eta}{\rho}=0 \quad \ldots \quad (3.1b)$$

$$-\frac{C_{,11}}{2C}+\left(\frac{C_{,1}}{2C}\right)^2-\frac{\eta}{C}\Gamma^1_{\underset{\vee}{14},1}-\left(\frac{\eta_{,1}}{C}+\frac{\eta^2}{C^2}\Gamma^1_{\underset{\vee}{14}}\right)\Gamma^1_{\underset{\vee}{14}}=0 \quad \ldots \quad (3.1c)$$

$$\Gamma^1_{\underset{\vee}{14},4}+\left(\frac{\eta_{,4}}{\eta}-\frac{C_{,4}}{C}\right)\Gamma^1_{\underset{\vee}{14}}=0 \quad \ldots \quad (3.1d)$$

$$C_{,1}+2\eta\Gamma^1_{\underset{\vee}{14}}=0 \quad \ldots \quad (3.1e)$$

$$\frac{C_{,11}}{2}+\frac{C_{,1}}{2\rho}-\frac{C^2_{,1}}{4C}+2\eta\Gamma^1_{\underset{\vee}{14},1}+\left[2\eta_{,1}+\frac{2\eta}{\rho}-\frac{3\eta C_{,1}}{2C}+\left(1-\frac{2\eta^2}{C}\right)\Gamma^1_{\underset{\vee}{14}}\right]\Gamma^1_{\underset{\vee}{14}}+\frac{\eta^2}{\rho^2}=0. \quad \ldots \quad (3.1f)$$

Integrating (3.1*a*) with respect to t, we obtain

$$C=K\eta^2 \quad \ldots \quad (3.2)$$

where K is a function of integration. Equation (3.1*d*) is identically satisfied by (3.2). Taking

$$K=1+k^2\rho^2 \quad \ldots \quad (3.3)$$

where k is a constant, we have

$$C=\eta^2(1+k^2\rho^2). \quad \ldots \quad (3.4)$$

Differentiating (3.4) with respect to ρ, we obtain

$$C_{,1}=2\eta\eta_{,1}(1+k^2\rho^2)+2\rho k^2\eta^2. \quad \ldots \quad (3.5)$$

Equations (3.1*b*), (3.1*e*) and (3.5) lead to the equation

$$\frac{\eta_{,1}}{\eta}+\frac{1}{\rho}=0. \quad \ldots \quad (3.6)$$

Integrating (3.6) with respect to ρ, we obtain

$$\eta=\frac{mF}{\rho} \quad \ldots \quad (3.7)$$

where m is a constant and F is a function of t. Now from (3.4) and (3.7) we obtain

$$C=F^2m^2k^2\left(1+\frac{m'^2}{\rho^2}\right) \quad \ldots \quad (3.8)$$

where

$$m'=\frac{1}{k}. \quad \ldots \quad (3.9)$$

By the coordinate transformation

$$\rho' = \rho, \quad \varphi' = \varphi, \quad z' = z, \quad t' = mk\int F\, dt \qquad \text{.. .. (3.10)}$$

we obtain

$$g'_{11} = \frac{g'_{22}}{\rho^2} = g'_{33} = -1, \quad g'_{44} = 1+\frac{m'^2}{\rho^2}, \quad g'_{\underset{\vee}{14}} = \frac{m'}{\rho}, \qquad \text{.. (3.11)}$$

where $g'_{\alpha\beta}$ is given by

$$g'_{\alpha\beta} = g_{\lambda\mu}\frac{\partial x^\lambda}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^\beta}. \qquad \text{.. .. .. (3.12)}$$

The solution given by (3.7) and (3.8) gives singularity all along the z-axis.

4. The Nature of Electromagnetic Field

The electromagnetic field tensor $F_{\lambda\mu}$ is the dual tensor of $\overset{\alpha\beta}{g^{\vee}}$ and is defined by the equation

$$F_{\lambda\mu} = \tfrac{1}{2}\epsilon_{\lambda\mu\alpha\beta}\sqrt{-g}\, \overset{\alpha\beta}{g^{\vee}} \qquad \text{.. .. .. .. (4.1)}$$

where $\epsilon_{\lambda\mu\alpha\beta}$ is the Levi-Civita tensor density. The non-vanishing component of $F_{\lambda\mu}$ is given by

$$F_{23} = -\frac{\rho\eta}{\sqrt{C-\eta^2}}. \qquad \text{.. .. .. .. (4.2)}$$

Also

$$F^{23} = -\frac{\eta}{\rho\sqrt{C-\eta^2}} \qquad \text{.. .. .. .. (4.3)}$$

where

$$F^{\lambda\mu} = \underline{g}^{\lambda\alpha}\underline{g}^{\mu\beta}F_{\alpha\beta}. \qquad \text{.. .. .. .. (4.4)}$$

The charge-current vector I^λ is given by

$$I^\lambda = F^{\lambda\mu}_{\ \ ;\mu}. \qquad \text{.. .. .. .. (4.5)}$$

Here we get

$$I^1 = I^2 = I^3 = I^4 = 0. \qquad \text{.. .. .. (4.6)}$$

Equation (4.6) suggests that there is no charge and current in the field taken into consideration. The magnitude of the magnetic intensity varies inversely as the distance from z-axis. The electromagnetic energy tensor E^μ_ν is defined by the equation

$$E^\mu_\nu = -F^{\mu\alpha}F_{\nu\alpha} + \tfrac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}. \qquad \text{.. .. .. (4.7)}$$

The non-vanishing components of E^μ_ν are given by

$$E_1^{\ 1} = -E_2^{\ 2} = -E_3^{\ 3} = E_4^{\ 4} = \frac{\eta^2}{2(C-\eta^2)}. \qquad \text{.. .. (4.8)}$$

Equations (3.7), (3.8) and (4.8) lead us to the conclusion that each of the non-vanishing components of E^μ_ν varies inversely as the square of the distance from the z-axis.

Finally, the metric of the space-time takes the form

$$ds^2 = -d\rho^2-\rho^2d\phi^2-dz^2+\left(1+\frac{m'^2}{\rho^2}\right)dt^2. \quad \text{(4.9)}$$

It shows that the magnetic field taken into consideration will be accompanied by a repulsive gravitational field. This idea should not appear as queer because the problem of negative masses is already being investigated.

This is an attempt to obtain an exact solution of the field equations in Einstein's unified field theory. In order to obtain an exact solution one has to make certain simplifying assumptions and once such a solution is possible and obtained, gradually more general solutions of exact nature can be attempted. From that point of view this attempt is important in itself.

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