LATTICE DOUBLE COVERINGS IN THE PLANE

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Here it is shown that the double covering constant $C_2(K)$ of a symmetric convex domain K is equal to half its covering constant $C(K)$. Examples of symmetric star domains T and S are given such that $C_2(T) = \frac{1}{2}\, C(T)$ and $C_2(S) > \frac{1}{2}\, C(S)$. The covering constant of the star domain S and all its maximal covering lattices are also determined.

§ 1. Let R_n be the n-dimensional Euclidean space. Let S be a closed set in R_n. A lattice Λ in R_n is said to provide an r-fold ($r \geqslant 1$) covering of R_n by S if every point of the space is contained in at least r translates of S through points of Λ. The r-covering constant of S is defined as

$$c_r(S) = \sup d(\Lambda)$$

where the suprimum is taken over all lattices Λ which provide an r-fold covering of R_n by S.

$c_1(S) = c(S)$ is the usual covering constant of S. It is easy to see that

$$c_r(S) \geqslant \frac{1}{r}\, c(S) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

for all natural numbers r.

Blundon (1957) determined $c_r(S)$ for $r = 2, 3, 4$ in case S is a circle with centre O. In particular he showed that equality holds in (1) for $r = 2$ and that there is strict inequality in (1) for $r = 3$ and 4. Few (1967) determined $c_2(K)$ when K is a sphere in R_3 with centre O and proved that strict inequality holds in (1) for $r = 2$.

In § 2 we extended Blundon's result to prove that $c_2(K) = \frac{1}{2}c(K)$ for all symmetric convex domains with centre O.

In §§ 3–5 we give examples of symmetrical star domains T and S such that

$$\text{(i)} \quad c_2(T) = \tfrac{1}{2}c(T)$$

and

$$\text{(ii)} \quad c_2(S) > \tfrac{1}{2}c(S).$$

The covering constant of T was determined by Hans (1970).

The covering constant of S and all maximal covering lattices of S will be determined in § 4. We shall also make use of these in another paper to prove that the result of Fary (1950) (*see* remark at the end of § 2) cannot be generalized to symmetrical star domains.

§ 2. In this section we prove the following theorem:

Theorem 1—If K is a closed symmetrical convex domain with centre O, then $c_2(K) = \frac{1}{2}\, c(K)$.

PROOF: First suppose that K is a strictly convex domain with centre O. Let Λ be a maximal covering lattice for K. The existence of such a lattice is well known. Then $d(\Lambda) = c\epsilon(K)$. Let A, B be a basis of Λ; consider the lattice Λ' generated by $A' = \frac{1}{2}A$ and $B' = B$. Then $d(\Lambda') = \frac{1}{2}d(\Lambda) = \frac{1}{2}c(K)$. Clearly Λ is the disjoint union of the grids Λ and $\Lambda + A'$ each of which provide a covering of the plane by K. Thus Λ gives a double covering of the plane by K. Therefore,

$$c_2(K) \geqslant d(\Lambda') = \tfrac{1}{2}c(K).$$

Let Λ be any lattice which gives a double covering by K. Since the origin O is covered at least twice, there exists $A \in \Lambda$, $A \neq O$ such that $O \in K + A$. It follows that $A \in K$, because K has centre O. We can suppose without loss of generality that A is a primitive lattice point. Since $A \in K \cap K + 2A$, K and $K + 2A$ intersect. Let S be a point of intersection of the boundaries of K and $K + 2A$. Then it is easy to verify that S does not belong to any $K + rA$, $r \neq 0$, 1, 2 and belongs to the common boundary of $K \cup K + 2A$. Thus there are points near S which do not belong to any set $K + rA$ other than $K + A$. On considering the double covering of these points we find that there exists a point B of Λ linearly independent of A such that $S \in K + B$. Therefore, the points O, $2A$, $B \in K + S$, i.e. the triangle Δ with vertices O, $2A$ and B is contained in $K + S$. Hence

$$\begin{aligned} d(\Lambda) \leqslant |\det(A, B)| &= \tfrac{1}{2}|\det(2A, B)| \\ &= \text{area of the triangle } \Delta \\ &\leqslant t(K) \end{aligned}$$

where $t(K)$ is the area of the largest triangle contained in K. Since clearly $2t(K) \leqslant c(K)$, we have

$$d(\Lambda) \leqslant \tfrac{1}{2}c(K).$$

Therefore,

$$c_2(K) \leqslant \tfrac{1}{2}c(K).$$

Hence

$$c_2(K) = \tfrac{1}{2}c(K).$$

For any convex domain K with centre O; for each ϵ, $0 < \epsilon < 1$, we can find a strictly convex domain K_ϵ with centre O such that

$$(1-\epsilon)K_\epsilon \subset K \subset (1+\epsilon)K_\epsilon.$$

Therefore,

$$c((1-\epsilon)K_\epsilon) \leqslant c(K) \leqslant c((1+\epsilon)K_\epsilon)$$

and

$$c_2((1-\epsilon)K_\epsilon) \leqslant c_2(K) \leqslant c_2((1+\epsilon)K_\epsilon)$$

i.e.

$$(1-\epsilon)^2\, c(K_\epsilon) \leqslant c(K) \leqslant (1+\epsilon)^2\, c(K_\epsilon)$$

and

$$(1-\epsilon)^2 c_2(K_\epsilon) \leqslant c_2(K) \leqslant (1+\epsilon)^2 c_2(K_\epsilon).$$

Since $c_2(K_\epsilon) = \frac{1}{2} c(K_\epsilon)$, we have

$$\begin{aligned} | c_2(K) - \tfrac{1}{2}c(K) | &\leqslant \{(1+\epsilon)^2-(1-\epsilon)^2\} . \tfrac{1}{2}c(K_\epsilon) \\ &= 2\epsilon\, c(K_\epsilon) \\ &\leqslant \frac{2\epsilon}{(1-\epsilon)^2} c(K). \end{aligned}$$

Since this is true for all ϵ $(0 < \epsilon < 1)$ we must have $c_2(K) = \frac{1}{2} c(K)$.

This completes the proof of Theorem 1.

Remark: Using the same technique we can give a simple proof of Fary's theorem (1950) for symmetrical convex domains, i.e. for any symmetrical convex domain K with centre O, we have $c(K) = 2t(K) =$ twice the area of the largest triangle contained in K-area of the largest symmetrical hexagon contained in K.

§ 3. Here we prove the following theorem.

Theorem 2—Let T be the symmetric star domain defined by

$$T : \begin{Bmatrix} \max \{ |x|, |y| \} < \frac{3}{2} \\ \min \{ |x|, |y| \} \leqslant 1 \end{Bmatrix}.$$

Then

$$c_2(T) = \tfrac{1}{2}c(T) = \tfrac{15}{4}.$$

PROOF: Hans (1970) proved that $c(T) = \frac{15}{2}$. We shall prove that $c_2(T) \leqslant \frac{15}{4}$. Since we always have $c_2(T) \geqslant \frac{1}{2} c(T) = \frac{15}{4}$, it will follow that $c_2(T) = \frac{15}{4}$.

Let Λ be any lattice which provides a double covering for T. Since the origin O must be covered twice there exists a point $A \neq O$ of Λ such that $A \in T$. By applying a suitable automorph of T if necessary we can suppose that A belongs to the shaded region shown in Fig. 2. We can suppose further

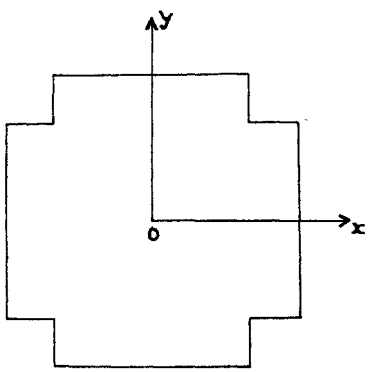

FIG. 1

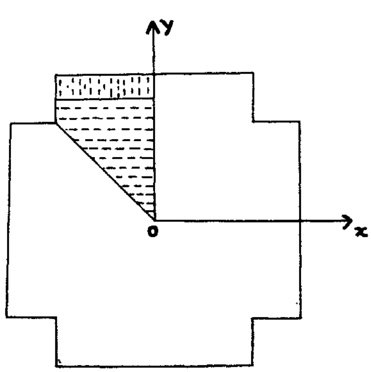

FIG. 2

that A is a primitive lattice point. Let $A = (a_1, a_2)$. Then

$$-a_1 \leqslant a_2 \leqslant \tfrac{3}{2},\ -1 \leqslant a_1 \leqslant 0.$$

We divide the discussion into the following cases:

$$\text{(I)}\quad -a_1 \leqslant a_2 \leqslant \tfrac{5}{4},\ -1 \leqslant a_1 \leqslant 0.$$

$$\text{(II)}\quad \tfrac{5}{4} < a_2 \leqslant \tfrac{3}{2},\ -1 \leqslant a_1 \leqslant 0.$$

Case I:
$$-1 \leqslant a_1 \leqslant 0,\ -a_1 \leqslant a_2 \leqslant \tfrac{5}{4}.$$

Consider the point $P = (2a_1+\frac{3}{2}, \frac{3}{2})$. Since $a_2 \leqslant \frac{5}{4}$, clearly P lies on the boundary of $T \cup T+2A$. There are points near P which are not double covered by the union of $T+mA$, m any integer, because if $m \leqslant -1$, then

$$|\tfrac{3}{2}-ma_2| = \tfrac{3}{2}-ma_2 \geqslant \tfrac{3}{2}+a_2 > \tfrac{3}{2} \text{ (since } a_2 > 0);$$

so that $P \notin T+mA$ for $m \leqslant -1$.

If $m \geqslant 3$, then

$$|2a_1+\tfrac{3}{2}-ma_1| = \tfrac{3}{2}-(m-2)\ a_1 \geqslant \tfrac{3}{2}-a_1 \geqslant \tfrac{3}{2}$$

with strict inequality unless $a_1 = 0$. Thus $P \notin T+mA$ for $m \geqslant 3$ unless $a_1 = 0$ in which case clearly P lies on the boundary of the union of $T+mA$.

Thus by considering the double covering of points near P we find that there exists $B \in \Lambda$ linearly independent of A such that $P \in T+B$. Let $B = (b_1, b_2)$. Therefore, we have either

$$\text{(i)}\quad |b_1-2a_1-\tfrac{3}{2}| \leqslant \tfrac{3}{2} \text{ and } |b_2-\tfrac{3}{2}| \leqslant 1,$$

or

$$\text{(ii)}\quad |b_1-2a_1-\tfrac{3}{2}| \leqslant 1 \text{ and } |b_2-\tfrac{3}{2}| \leqslant \tfrac{3}{2}.$$

Case I (i)—In this case we have

$$2a_1 \leqslant b_1 \leqslant 2a_1+3 \text{ and } \tfrac{1}{2} \leqslant b_2 \leqslant \tfrac{5}{2}.$$

Therefore,

$$-3 = -2\,.\,\tfrac{3}{2} \leqslant 2a_1a_2 \leqslant b_1a_2-b_2a_1 \leqslant (2a_1+3)a_2-\tfrac{5}{2}a_1$$
$$\leqslant \tfrac{5}{4}(2a_1+3)-\tfrac{5}{2}a_1 = \tfrac{15}{4} \text{ (since } a_2 \leqslant \tfrac{5}{4}).$$

Hence

$$d(\Lambda) \leqslant |b_1a_2-b_2a_1| \leqslant \tfrac{15}{4}.$$

Case I (ii)—In this case we have

$$2a_1+\tfrac{1}{2} \leqslant b_1 \leqslant 2a_1+\tfrac{5}{2} \text{ and } 0 \leqslant b_2 \leqslant 3.$$

Therefore,

$$-\tfrac{15}{8} \leqslant -\tfrac{3}{2}a_2 \leqslant (2a_1+\tfrac{1}{2})a_2 \leqslant b_1a_2-b_2a_1 \leqslant (2a_1+\tfrac{5}{2})a_2-3a_1$$
$$\leqslant \tfrac{5}{4}(2a_1+\tfrac{5}{2})-3a_1 = \tfrac{25}{8}-\frac{a_1}{2} \leqslant \tfrac{25}{8}+\tfrac{1}{2} < \tfrac{15}{4}.$$

Hence

$$d(\Lambda) \leqslant |b_1a_2-b_2a_1| < \tfrac{15}{4}.$$

Thus the result is true in Case I.

Case II—
$$-1 \leqslant a_1 \leqslant 0,\ \tfrac{5}{4} < a_2 \leqslant \tfrac{3}{2}.$$

Consider the point $Q = (2a_1+1, \frac{3}{2})$. We claim that $Q \notin T+mA$, for integers $m \neq 0, 1, 2$. For if $m \leqslant -1$, then

$$|\tfrac{3}{2}-ma_2| = \tfrac{3}{2}-ma_2 \geqslant \tfrac{3}{2}+a_2 > \tfrac{3}{2} \text{ (since } a_2 > 0)$$

and if $m \geqslant 3$, then

$$|\tfrac{3}{2}-ma_2| = ma_2-\tfrac{3}{2} \geqslant 3a_2-\tfrac{3}{2} \geqslant \tfrac{15}{4}-\tfrac{3}{2} > \tfrac{3}{2} \text{ (since } a_2 > \tfrac{5}{4}).$$

Since $a_2 > 5/4$, i.e. $2a_2-1 > \frac{3}{2}$; P belongs to the boundary of $T \cup T+2A$ and there are points near P which belong only to possibly $T+A$.

Thus by considering the double covering of these points we find that there exists a point $B \in \Lambda$, B linearly independent of A such that $P \in T+B$. Let $B = (b_1, b_2)$. Therefore we have either

$$\text{(i)} \quad |b_1-2a_1-1| \leqslant \tfrac{3}{2} \text{ and } |b_2-\tfrac{3}{2}| \leqslant 1$$

or

$$\text{(ii)} \quad |b_1-2a_1-1| \leqslant 1 \text{ and } |b_2-\tfrac{3}{2}| \leqslant \tfrac{3}{2}.$$

Case II (i)—In this case we have

$$2a_1-\tfrac{1}{2} \leqslant b_1 \leqslant 2a_1+\tfrac{5}{2} \text{ and } \tfrac{1}{2} \leqslant b_2 < \tfrac{5}{2}.$$

Therefore,

$$\begin{aligned} b_1a_2-b_2a_1 &\leqslant (2a_1+\tfrac{5}{2})a_2-\tfrac{5}{2}a_1 \\ &\leqslant \tfrac{3}{2}(2a_1+\tfrac{5}{2})-\tfrac{5}{2}a_1 \\ &= \tfrac{15}{4}+\tfrac{1}{2}a_1 \leqslant \tfrac{15}{4} \text{ (since } -1 \leqslant a_1 \leqslant 0). \end{aligned}$$

It is easy to verify that $b_1a_2-b_2a_1 \geqslant -\frac{15}{4}$.
Hence

$$d(\Lambda) \leqslant \tfrac{15}{4}.$$

Case II (ii)—In this case we have

$$2a_1 \leqslant b_1 \leqslant 2a_1+2 \text{ and } 0 \leqslant b_2 \leqslant 3.$$

Therefore,

$$-3 \leqslant 2a_1a_2 \leqslant b_1a_2-b_2a_1 \leqslant (2a_1+2)a_2-3a_1 \leqslant \tfrac{3}{2}(2a_1+2)-3a_1 = 3$$

so that

$$d(\Lambda) \leqslant 3 < \tfrac{15}{4}.$$

Thus in all cases $d(\Lambda) \leqslant \frac{15}{4}$. Hence $c_2(T) \leqslant \frac{15}{4}$. This completes the proof of Theorem 2.

§ 4. Let $P_1 = (3, 0)$, $Q_1 = (15, 3)$, $R_1 = (15, 15)$, $Q_2 = (3, 15)$, $P_2 = (0, 3)$, $Q_3 = (-3, 15)$, $R_2 = (-15, 15)$, $Q_4 = (-15, 3)$, $P_3 = (-3, 0) = -P_1$.

Let S be the set bounded by the line segments P_1Q_1, Q_1R_1, R_1Q_2, Q_2P_2, P_2Q_3, Q_3R_2, R_2Q_4, Q_4P_3 and their reflection in the origin (*see* Fig. 3). Then S is a star domain symmetrical about the origin, about the axes and about the lines $y = x$ and $y = -x$.

We denote the segments P_1Q_1, P_2Q_2, P_2Q_3, P_3Q_4 and the corresponding lines by the same symbols l_1, l_2, l_3, l_4 respectively.

In this section we determine the covering constant and all maximal covering lattices of S. More precisely we prove the following.

Theorem 3—Let S be the symmetric star domain described above. Then

$$c(S) = 491{\cdot}04.$$

The maximal covering lattices of S are of the type $\Omega\Gamma$, where Ω is an automorph of S and Γ is the lattice generated by

$$A = \left(-\frac{66}{5}, \frac{66}{5}\right) \text{ and } B = \left(12, \frac{126}{5}\right).$$

PROOF: Let Λ be any covering lattice of S. Consider the point $P_2 = (0, 3)$. It must be covered by a domain $S+A$ for some point $A \in \Lambda$, $A \neq O$. Then $A \in S+P_2$. On replacing Λ by a lattice $\Omega\Lambda$ where Ω is a suitable automorph of S, we can suppose that A lies in the region

$$\mathfrak{R} = \{(x, y) | -15 \leqslant x \leqslant 0,\ -x \leqslant y \leqslant 18\} \text{ (shaded region in Fig. 4)}$$

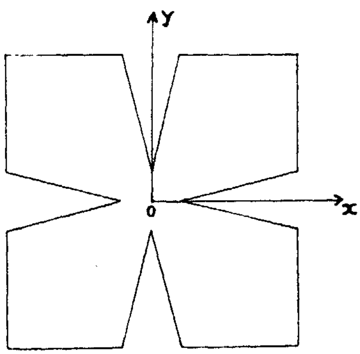

FIG. 3

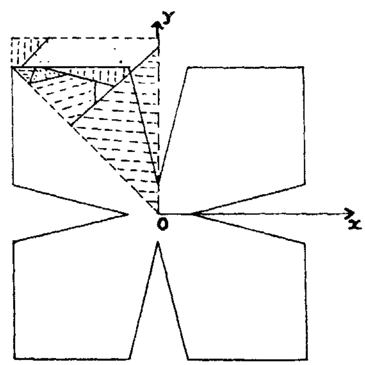

FIG. 4

We note that with this choice of A, P_2 may not belong to $S+A$. If A lies in $\mathfrak{R}$, then the line segment OA also lies in $\mathfrak{R}$, so we can suppose without loss of generality that A is a primitive point of Λ. Let $A = (a_1, a_2)$. Then

$$-15 \leqslant a_1 \leqslant 0,\ -a_1 \leqslant a_2 \leqslant 18.$$

The following Lemma is easy to prove:

Lemma A—Let A and C be linearly independent points of Λ with $A = (a_1, a_2)$ in the region $\mathfrak{R}$. Then $d(\Lambda) < 491$ if any one of the following three conditions is satisfied:

(i) $a_2 \leqslant 15$, C lies below OA and a point P above the line OA is covered by the domain $S+C$.

(ii) $15 < a_2 \leqslant 18$, C lies below OA and a point P on or above the line through $Q = (a_1+3, a_2)$ parallel to OA is covered by the domain $S+C$.

(iii) If there exists a point $P = (x, y)$ such that $a_1+3 \leqslant x \leqslant a_1+15$, $3 \leqslant y \leqslant 18$; P lies in $S+C$ and A, C do not generate the lattice Λ.

We shall consider suitable points not covered by the union of the sets $S+rA$, r any integer, so that in order to cover these points we get a point B of Λ independent of A such that $d(\Lambda) \leqslant |\det(A, B)| \leqslant 491{\cdot}04$ with equality only if Λ is the lattice Γ described in the theorem.

We divide the discussion into two parts:

(I) $0 < a_2 \leqslant 15$;

(II) $15 < a_2 \leqslant 18$.

Case I

Let $E = (2a_1+15,\ 15)$;

$F = (63+a_1-4a_2,\ 15)$, the point of intersection of the line l_1+A with $y = 15$;

$G = \left(2a_1+15,\ a_2+3+\frac{a_1}{4}\right)$, the point of intersection of the line l_1+A with $x = 2a_1+15$;

$$H = \frac{15+a_1-4a_2}{71},\ \frac{16a_2-4a_1-9}{17},$$

the point of intersection of the line l_1+A with l_3;

$$K = \frac{a_1+4a_2-9}{17},\ \frac{4a_1+16a_2+15}{17},$$

the point of intersection of line l_4+A with line l_2;

$L = (a_1+15,\ 4a_1+63)$, the point of intersection of $x = a_1+15$ with l_2.

We have a number of subcases to consider depending upon the position of A and the relative position of the points described above.

[I(i)] $9a_2-8a_1 \leqslant 153$.

[I(ii)] $9a_2-8a_1 > 153$ and $a_1+4a_2 \geqslant 48$, i.e. E lies to the right of F.

[I(iii)] $9a_2-8a_1 > 153$, $a_1+4a_2 < 48$, i.e. F lies to the right of E and $a_1 \geqslant -6{\cdot}3$.

[I(iv)] $9a_2-8a_1 > 153$, $a_1+4a_2 < 48$, $a_1 < -6{\cdot}3$ and $a_1-4a_2+66 > 0$, i.e. F lies to the right of $(-3, 15)$.

[I(v)] $9a_2-8a_1 > 153$, $a_1+4a_2 < 48$, $a_1 < -6{\cdot}3$, $a_1-4a_2+66 \leqslant 0$, i.e. F lies to the left of $(-3, 15)$. In this case we have two further subcases:

[I(v*a*)] $4a_1-a_2+66 \geqslant 0$; in this case the line segment $-l_4+A$ meets l_2 in K.

[I(v*b*)] $4a_1-a_2+66 < 0$; in this case the line segment $-l_4+A$ does not meet l_2.

The corresponding sub-regions of $\mathcal{R}$ are shown in Fig. 4.

Now we discuss the various subcases:

Case I(i): $9a_2-8a_2 \leqslant 153$—On considering the covering of the points near $D=(a_1+15,\ 15)$ we find that there exists a point $B=(b_1,\ b_2)$ of Λ linearly independent of A such that $D \in S+B$, i.e. $B \in S+D$. Therefore,

$$a_1 \leqslant b_1 \leqslant a_1+30, \quad 0 \leqslant b_2 \leqslant 30.$$

If B is below OA, by Lemma A(i) we have $d(\Lambda) < 491$. If B is above OA, then

$$d(\Lambda) \leqslant |\det(A,\ B)| = b_1a_2-b_2a_1 \leqslant (a_1+30)a_2-30a_1 = f(a_1,\ a_2).$$

If $a_2 \leqslant 9$, then

$$f(a_1,\ a_2) \leqslant 9(a_1+30)-30a_1 = 270-21a_1 \leqslant 459 \text{ since } -a_1 \leqslant a_2 \leqslant 9.$$

If $a_2 > 9$, then

$$\begin{aligned} f(a_1,\ a_2) &= 30a_2-a_1(30-a_2) \\ &\leqslant 30a_2+\tfrac{1}{8}(153-9a_2)(30-a_2) \\ &= \frac{9}{8}a_2^2-\frac{183}{8}a_2+\frac{2295}{4} = g(a_2) \text{ (say)} \\ &\leqslant \max(g(9),\ g(15)) < 491. \end{aligned}$$

Thus in this case $d(\Lambda) < 491$.

Case I(ii): $9a_2-8a_1 > 153$ and $a_1+4a_2 \geqslant 48$—In this case E lies to the right of F. On considering the covering of E we must have a point $B=(b_1,\ b_2)$ of Λ linearly independent of A such that E lies in $S+B$; so that

$$2a_1 \leqslant b_1 \leqslant 2a_1+30, \quad 0 \leqslant b_2 \leqslant 30.$$

If B is below OA, by Lemma A(i) we have $d(\Lambda) < 491$. If B is above OA, then

$$\begin{aligned} d(\Lambda) \leqslant |\det(A,\ B)| = b_1a_2-b_2a_1 &\leqslant (2a_1+30)a_2-30a_1 \\ &\leqslant (2a_1+30)15-30a_1 \\ &= 450 < 491. \end{aligned}$$

and we are through in this case also.

Case I(iii): $9a_2-8a_1 > 153$, $a_1+4a_2 < 48$; $a_1 \geqslant -6{\cdot}3$—In this case F lies to the right of E. On considering the covering of F we must have a point $B=(b_1,\ b_2)$ of Λ linearly independently of A such that F lies in $S+B$. Therefore,

$$48+a_1-4a_2 \leqslant b_1 \leqslant 78+a_1-4a_2, \ 0 \leqslant b_2 \leqslant 30.$$

If B lies below OA, then by Lemma A(i) we have $d(\Lambda) < 491$. If B lies above OA, then

$$d(\Lambda) \leqslant |\det(A,\ B)| = b_1a_2-b_2a_1 \leqslant (78+a_1-4a_2)a_2-30a_1 < 491$$

as can be easily verified under the given conditions.

Remark: Now let $9a_2-8a_1 > 153$, $a_1+4a_2 < 48$, $a_1 < -6{\cdot}3$.
In this case it is easy to see that G lies above the line l_2.

Case I(iv): $9a_2-8a_1>153$, $a_1+4a_2<48$, $a_1<-6\cdot 3$ and $a_1-4a_2+66>0$—In this case F is to the right of $Q_3=(-3, 15)$ and the line segment l_1+A meets the line segment l_3 in H. We have two subcases to consider:

I(iv) (a): $H \in \mathfrak{B}+2A$,

I(iv) (b): $H \notin \mathfrak{B}+2A$,

where $\mathfrak{B}$: $|x| \leqslant 15$, $|y| \leqslant 15$.

Case I(iva): On considering the covering of the point G, we find that there exists a point $B=(b_1, b_2)$ of Λ, B linearly independent of A such that $G \in S+B$. Therefore

$$2a_1 \leqslant b_1 \leqslant 2a_1+30$$

$$a_2-12+\frac{a_1}{4} \leqslant b_2 \leqslant a_2+18+\frac{a_1}{4} < 30 \text{ (since } a_1+4a_2<48).$$

If B lies below OA, then by Lemma A(i) we have $d(\Lambda)<491$. If B lies above OA, then

$$d(\Lambda) \leqslant b_1a_2-b_2a_1 \leqslant (2a_1+30)a_2-30a_1$$
$$\leqslant 15(2a_1+30)-30a_1=450<491.$$

Case I(ivb): In this case consider the points H and K which belong to the boundary of $S \cup S+A$ and to no other $S+rA$ (r integer). Both these points lie above OA. On considering their covering we find that there exist points B and C of Λ linearly independent of A such that

$$K \in S+B,\ H \in S+C.$$

We can suppose that (i) B and C are above OA, (ii) A, B generate and (iii) A, C generate Λ, for otherwise by Lemma A, $d(\Lambda)<491$.

Thus we can write $C=B+nA$ for some integer n. Let $B=(b_1, b_2)$. Then $b_1a_2-b_2a_1>0$ and

$$b_1 \leqslant \min\left(\frac{a_1+4a_2+246}{17}, \frac{a_1-4a_2+270}{17}-na_1\right)$$

$$b_2 \leqslant \min\left(\frac{4a_1+16a_2+270}{17}, \frac{16a_2-4a_1+246}{17}-na_2\right).$$

If $n \leqslant 0$, we have

$$b_1 \leqslant \frac{a_1-4a_2+270}{17},\ b_2 \leqslant \frac{4a_1+16a_2+270}{17};$$

so that

$$d(\Lambda)=b_1a_2-b_2a_1 \leqslant \tfrac{1}{17}\{a_2(a_1-4a_2+270)-a_1(4a_1+16a_2+270)\}$$
$$=f(a_1, a_2) \quad \text{(say)}.$$

Since

$$\frac{\partial f}{\partial a_2} > 0 \text{ for } a_2 < \frac{a_1+66}{4}, \text{ we have}$$

$$d(\Lambda) < f\left(a_1, \frac{a_1+66}{4}\right) = \tfrac{1}{17}\{51(a_1+66)-a_1(8a_1+534)\}$$

$$= g(a_1) \quad \text{(say)}.$$

Since $g(a_1)$ is decreasing and $a_1 \geqslant -\frac{66}{5}$, we have

$$d(\Lambda) < g(-\tfrac{66}{5}) = 491{\cdot}04.$$

If $n \geqslant 1$, then we have

$$b_1 < \frac{a_1+4a_2+246}{17}$$

$$b_2 \leqslant \frac{16a_2-4a_1+246}{17} - a_2 = \frac{246-4a_1-a_2}{17}.$$

Therefore,

$$d(\Lambda) = b_1a_2-b_2a_1 \leqslant \tfrac{1}{17}\{a_2(a_1+4a_2+246)-a_1(246-4a_1-a_2)\} < 491$$

as can be easily verified.

This settles the case I(iv).

Case I(v): $9a_2-8a_1 > 153$, $a_1+4a_2 < 48$, $a_1 < -6{\cdot}3$, $a_1-4a_2+66 \leqslant 0$.— In this case F is to the left of $Q_3 = (-3, 15)$ and belongs to the boundary of $S \cup S+A$ and to no other $S+rA$ for integers r.

Case I(va): $4a_1-a_2+66 \geqslant 0$.—In this case the line segment $-l_4+A$ meets l_2 in K. Both F and K are above OA. On considering their covering we find that there exist points B and C of Λ linearly independent of A such that

$$K \in S+B, \quad F \in S+C.$$

We can suppose that (i) B and C are above OA; (ii) A, B generate Λ; (iii) A, C generate Λ, for otherwise by Lemma A, $d(\Lambda) < 491$.
Thus we can write $C = B+nA$ for some integer n.
Therefore,

$$b_1 \leqslant \min\left(\frac{a_1+4a_2+246}{17}, 78-4a_2+a_1-na_1\right)$$

$$b_2 \leqslant \min\left(\frac{4a_1+16a_2+270}{17}, 30-na_2\right).$$

If $n \leqslant 0$, we have

$$b_1 \leqslant 78-4a_2+a_1, \quad b_2 \leqslant \frac{4a_1+16a_2+270}{17}.$$

Therefore,

$$d(\Lambda) = b_1a_2-b_2a_1 < (78+a_1-4a_2)a_2 - \frac{a_1}{17}(4a_1+16a_2+270)$$

$$= f(a_1, a_2) \quad \text{(say)}.$$

It is easy to verify that for $a_2 \geqslant \frac{a_1+66}{4}$, $\frac{\partial f}{\partial a_2} < 0$. Thus we have

$$d(\Lambda) \leqslant f\left(a_1, \frac{a_1+66}{4}\right) = 3(a_1+66) - \frac{a_1(8a_1+534)}{17}$$
$$= g(a_1) \quad \text{(say)}.$$

Since in this case $a_1 \geqslant -\frac{66}{5}$ and $g(a_1)$ is decreasing, we have

$$d(\Lambda) \leqslant g(-\tfrac{66}{5}) = 491{\cdot}04.$$

We have strict inequality unless $a_1 = -\frac{66}{5}$, $a_2 = \frac{66}{5}$, $n = 0$, $b_1 = 12$ and $b_2 = \frac{126}{5}$, i.e. $\Lambda = \Gamma$.

If $n \geqslant 1$, we have

$$b_1 \leqslant \frac{a_1+4a_2+246}{17}, \quad b_2 \leqslant 30-na_2 \leqslant 30-a_2.$$

Therefore,

$$d(\Lambda) = b_1a_2-b_2a_1 \leqslant a_2\left(\frac{a_1+4a_2+246}{17}\right)-a_1(30-a_2)$$
$$\leqslant 450$$

as can be easily verified.

Case I(vb): Since in this case $4a_1-a_2+66 < 0$, $a_1+a_2 \geqslant 0$ we have $a_2 > \frac{66}{5}$ and $a_1 < -\frac{51}{4}$. In this case the line segment $-l_4+A$ does not meet l_2. Instead of K we consider the point $L = (a_1+15, 4a_1+63)$ which is the point of intersection of $x = a_1+15$ with l_2. L lies on the boundary of $S \cup S+A$ and to no other $S+rA$ for integer r. On considering the covering of the points F and L and using Lemma A we find that either $d(\Lambda) < 491$ or there exists a point $B = (b_1, b_2)$ of Λ such that

(i) A, B generate Λ.
(ii) B lies above OA.
(iii) $L \in S+B$, $F \in S+B+nA$ for some integer n.

Therefore,

$$b_1 \leqslant \min\,(a_1+30,\ 78-4a_2+a_1-na_1)$$
$$b_2 \leqslant \min\,(78+4a_1,\ 30-na_2).$$

If $n \leqslant 0$, we have

$$b_1 \leqslant 78-4a_2+a_1, \quad b_2 \leqslant 78+4a_1.$$

Hence

$$d(\Lambda) = b_1a_2-b_2a_1 \leqslant (78-4a_2+a_1)a_2-a_1(78+4a_1)$$
$$= f(a_1, a_2) \quad \text{(say)}.$$

Since $\frac{\partial f}{\partial a_1} > 0$ for $4a_1-a_2+66 < 0$, we have

$$d(\Lambda) < f\left(\frac{a_2-66}{4}, a_2\right) = \left(78-\frac{15a_2+66}{4}\right)a_2-\frac{(a_2+12)(a_2-66)}{4}$$
$$= g(a_2), \text{ say}.$$
$$< g(\tfrac{66}{5}) = 491{\cdot}04$$

since $g(a_2)$ is a decreasing function of a_2 for $a_2 > \frac{66}{5}$.

If $n \geqslant 1$, then

$$b_1 \leqslant 30+a_1,\ b_2 \leqslant 30-na_2 \leqslant 30-a_2,$$

so that

$$d(\Lambda) = b_1a_2-b_2a_1 \leqslant (30+a_1)a_2-a_1(30-a_2) < 491$$

as can be easily verified. Thus in this case $d(\Lambda) < 491{\cdot}04$.

This completes the discussion of Case I.

Case II—

$$15 < a_2 \leqslant 18,\ -15 \leqslant a_1 \leqslant 0.$$

Consider the covering of the point $Q = (a_1+3,\ a_2)$. We must have a point $B = (b_1,\ b_2)$ of Λ linearly independent of A such that $Q \in S+B$, so that

$$a_1-12 \leqslant b_1 \leqslant a_1+18$$
$$a_2-15 \leqslant b_2 \leqslant a_2+15.$$

If B lies below OA, by Lemma A(ii) we have $d(\Lambda) < 491$. If B lies above OA but A, B do not generate Λ again by Lemma A(iii) we have $d(\Lambda) < 491$. Therefore, we can suppose that B lies above OA and A, B generate Λ. We have a number of subcases to consider

II(i): $18a_2-15a_1 \leqslant 491$.
II(ii): $18a_2-15a_1 > 491$ and $b_2 \leqslant 30$.
II(iii): $18a_2-15a_1 > 491$ and $30 < b_2 \leqslant a_2+15$.

Now we discuss the various cases.

Case II(i): $18a_2-15a_1 \leqslant 491$.—In this case we have

$$\begin{aligned} d(\Lambda) = b_1b_2-b_2a_1 &\leqslant (a_1+18)a_2-a_1(a_2+15) \\ &= 18a_2-15a_1 \\ &\leqslant 491. \end{aligned}$$

Case II(ii): $18a_2-15a_1 > 491$, $b_2 \leqslant 30$.—We have the following three subcases:

(a) $a_1 \geqslant -\frac{27}{2}$, then $$\begin{aligned} d(\Lambda) = b_1a_2-b_2a_1 &\leqslant (a_1+18)a_2-30a_1 \\ &\leqslant 18(a_1+18)-30a_1 \\ &= 324-12a_1 \\ &\leqslant 486. \end{aligned}$$

(b) $a_1 < -\frac{27}{2}$, $b_2 \leqslant 4a_1+78$.

Then

$$\begin{aligned} d(\Lambda) = b_1a_2-b_2a_1 &\leqslant (a_1+18)a_2-a_1(4a_1+78) \\ &\leqslant 18(a_1+18)-a_1(4a_1+78) \\ &= 324-4a_1(a_1+15) \\ &\leqslant 324+54(15-\tfrac{27}{2}) \\ &= 405. \end{aligned}$$

(c) $a_1 < -\frac{27}{2}$, $b_2 > 4a_1+78$.

In this case the point $N = (a_1+15,\ b_2-15)$ lies above the line l_2. On considering the covering of this point we find that it must be covered by a translate of S through a lattice point C other than rA or B. If C lies below OA or A, C do not generate the lattice, by Lemma A, we have $d(\Lambda) < 491$. Otherwise, we must have $C = B+nA$ for some integer $n \neq 0$; since $C \neq B$. Therefore,

$$a_1 \leqslant b_1+na_1 \leqslant a_1+30$$

$$b_2-30 \leqslant b_2+na_2 \leqslant b_2$$

i.e.

$$-30 \leqslant na_2 \leqslant 0.$$

Since $a_2 > 15$ and $n \neq 0$, this is only possible if $n = -1$. Then $b_1 \leqslant 2a_1+30$, $b_2 \leqslant 30$. Thus

$$\begin{aligned} d(\Lambda) = b_1a_2-b_2a_1 &\leqslant 2(a_1+30)a_2-30a_1 \\ &\leqslant 18(2a_1+30)-30a_1 \\ &= 540+6a_1 \\ &< 540-81 < 491. \end{aligned}$$

Thus in all subcases $d(\Lambda) < 491$.

Case II(iii): $18a_2-15a_1 > 491$, $30 < b_2 \leqslant a_2+15$.

Consider the point $M = (a_1+4a_2+63-4b_2,\ b_2-15)$, the point of intersection of $-l_4+A$ with $y = b-15$. On considering the covering of points near this point we find that M must be covered by a translate of S through a lattice point C other than rA or B. If C lies below OA or A and C do not generate Λ, by Lemma A we have $d(\Lambda) < 491$. Otherwise we must have $C = B+nA$ for some integer $n \neq 0$, since $C \neq B$. Since $M \in S+B+nA$, we have

$$b_1+na_1 \leqslant 4a_2+78+a_1-4b_2$$

$$-30+b_2 \leqslant b_2+na_2 \leqslant b_2$$

i.e.

$$-30 \leqslant na_2 \leqslant 0.$$

Since $n \neq 0$ and $a_2 > 15$ we must have $n = -1$. Therefore, we have

$$b_1 \leqslant 4a_2+78+2a_1-4b_2$$

$$30 < b_2 \leqslant a_2+15.$$

Thus

$$\begin{aligned} d(\Lambda) = b_1a_2-b_2a_1 &\leqslant (4a_2+78+2a_1-4b_2)a_2-b_2a_1 \\ &= f(a_1,\ a_2,\ b_2), \quad \text{say}. \end{aligned}$$

Since clearly $\dfrac{\partial f}{\partial b} < 0$ for $a_2 > 15$ and $b_2 > 30$, we have

$$\begin{aligned} d(\Lambda) &< f(a_1,\ a_2,\ 30) \\ &= (4a_2+2a_1-42)a_2-30a_1 < 491 \end{aligned}$$

as can easily be verified.

Thus we have proved that if Λ is a covering lattice for S, then $d(\Lambda) \leqslant 491{\cdot}04$ and strict inequality holds unless Λ is of the type $\Omega\Gamma$ where Ω is an automorph of S and Γ is the lattice generated by

$$A = \left(-\frac{66}{5}, \frac{66}{5}\right),\ B = \left(12, \frac{126}{5}\right).$$

We claim that Γ is a covering lattice for S. For this it suffices to prove that triangle OAB is covered by translates of S through points of Γ. It is easy to see that triangle OAB is covered by S, $S+A$, $S+B$ as is clear from Fig. 5. Hence $C(S) = 491{\cdot}04$ and maximal covering lattices of S are as stated in the theorem.

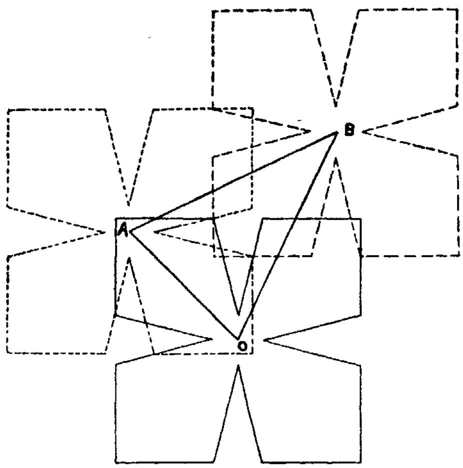

Fig. 5

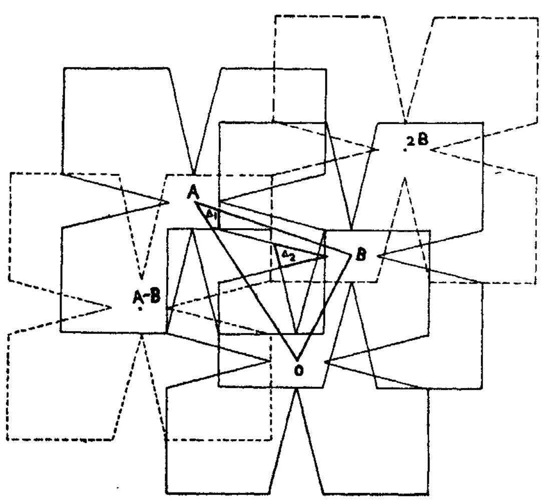

Fig. 6

§ 5. In this section we prove the following theorem.

Theorem 4—Let S be the star domain shown in Fig. 3. Then

$$C_2(S) \geqslant 252 > \tfrac{1}{2}C(S).$$

PROOF: Since by Theorem 3, $C(S) = 491{\cdot}04 < 504$, it suffices to prove that $C_2(S) \geqslant 252$. Consider the lattice Λ generated by $A = (-12, 18)$ and $B = (6, 12)$. Then $d(\Lambda) = 252$. We claim that Λ is a double covering lattice for S. For this it suffices to prove that triangle OAB is covered twice by the translates of S. The triangle OAB is covered at least once by S, $S+A$ and $S+B$ and the only portions which are not double covered by these are the regions Δ_1 and Δ_2 shown in Fig. 6. These are covered second time by $S+2B$ and $S+A-B$ respectively. Therefore,

$$C_2(S) \geqslant d(\Lambda) = 252.$$

This completes the proof of Theorem 4.

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