LATTICE DOUBLE PACKINGS IN THE PLANE

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It is proved that the lattice double packing density of a symmetrical convex domain is twice its lattice packing density. An example is given to show that this result cannot be extended to the class of symmetrical star domains.

§ **1**. Let R_n be the n-dimensional Euclidean space. Let S be an open set in R_n with volume $V(S)$. A lattice Λ in R_n is said to be a k-fold packing for S if no point of the space R_n is contained in more than k translates of S by points of Λ. In particular if $k = 1$, Λ is a packing lattice for S in the usual sense. The density of the best k-fold lattice packing of S is defined by

$$\delta_k(S) = \sup \frac{V(S)}{d(\Lambda)}$$

where $d(\Lambda)$ is the determinant of Λ and the suprimum is taken over all lattices Λ which provide a k-fold packing for S. For $k = 1$, $\delta_1(S) = \delta(S)$ is the usual density of the best lattice packing of S. It is easy to show that

$$\delta_k(S) \geqslant k\delta(S). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

If S is a circle with centre O, Heppes (1959) proved that equality holds in (1) if and only if $k \leqslant 4$. Blundon (1963) obtained $\delta_5(S)$ and $\delta_6(S)$ for the circle. For a sphere in three dimensional space $\delta_2(S)$ was determined by Few and Kangasabapathy (1969).

In § 2 we extend the result of Heppes to all symmetrical convex domains in R_2 for lattice double packings. In § 3 we give an example to show that this result cannot be extended to the class of symmetrical star domains.

§ **2**. Let S be an open set in R_2. A lattice Λ is said to be admissible for S (S-admissible) if Λ has no point other than the origin in S. The critical determinant $\Delta(S)$ of S is defined as the infimum of the determinants of all admissible lattices for S. It is well known that Λ is a packing lattice for S if and only if Λ is admissible for the difference set $D(S)$ of S. It follows that

$$\delta(S) = \frac{a(S)}{\Delta(D(S))}.$$

where $a(S)$ is the area of S. If K is a convex domain with centre O, then $D(K) = 2K$, hence $\delta(K) = \dfrac{a(K)}{4\Delta(K)}$.

We prove the following theorem:

Theorem 1—Let K be a symmetrical convex domain with centre O. Then

$$\delta_2(K) = 2\delta(K).$$

PROOF: In view of (1) it remains to prove that $\delta_2(K) \leqslant 2\,\delta(K)$. Since $\delta(K) = \dfrac{a(K)}{4\Delta(K)}$, it suffices to prove that if Λ is any lattice which provides a lattice double packing for K, then $d(\Lambda) \geqslant 2\,\Delta(K)$.

Let Λ be a double packing lattice for K. Let $f(X)$ be the gauge function of K so that K is given by $f(X) < 1$.

For any point A of Λ other than O, we must have $f(A) \geqslant 1$; for if $f(A) < 1$, then O belongs to the three sets K, $K+A$ and $K-A$.

If $f(A) \geqslant 2$ for all points of Λ other than the origin, then Λ is admissible for $2K$ which implies that

$$d(\Lambda) \geqslant \Delta(2K) = 4\Delta(K)$$

and the assertion follows.

Suppose that there is an $A \in \Lambda$, $A \neq O$ such that $f(A) < 2$. Among all such lattice points we choose A such that $f(A)$ is least. Thus $1 \leqslant f(A) < 2$ and $1 \leqslant f(A) \leqslant f(P)$ for all $P \in \Lambda, P \neq O$. Then A is clearly a primitive point of Λ. In case O, $\pm A$ are the only points of Λ in $2K$, then the lattice Λ' generated by $2A$ and B, where A, B is a basis of Λ, is admissible for $2K$. Therefore,

$$d(\Lambda) = \tfrac{1}{2}d(\Lambda') \geqslant \tfrac{1}{2}\Delta(2K) = 2\Delta(K)$$

and the assertion follows in this case.

Suppose that there exists a point B of Λ other than O, $\pm A$ in $2K$. Then $f(B) < 2$. Since $f(nA) = |n|\,f(A) \geqslant |n| \geqslant 2$ if $n \neq 0, \pm 1$ is an integer; B is linearly independent of A. We claim that A, B generate Λ. Let A, C be a basis of Λ. Then $B = mA+nC$ for some integers m, n; $n \neq 0$. On replacing C by $-C$ if necessary we can suppose that $n \geqslant 1$. We have

$$1 \leqslant f(A) \leqslant f(B) = f(mA+nC) < 2.$$

so that

$$f\left(\frac{m}{n}A+C\right) < \frac{2}{n}.$$

Choose an integer k such that $\left|\dfrac{m}{n}-k\right| \leqslant \tfrac{1}{2}$.

Therefore,

$$\begin{aligned} f(A) \leqslant f(kA+C) &= f\left(\frac{m}{n}A+C+\left(k-\frac{m}{n}\right)A\right) \\ &\leqslant f\left(\frac{m}{n}A+C\right)+\left|\frac{m}{n}-k\right| f(A) \\ &< \frac{2}{n}+\tfrac{1}{2}f(A) \end{aligned}$$

i.e. $f(A) < \frac{4}{n}$, which is a contradiction to $f(A) \geqslant 1$ if $n \geqslant 4$.

If $n = 3$, we have $\left|k - \frac{m}{3}\right| \leqslant \frac{1}{3}$, so that

$$f(A) \leqslant f(kA+C) < \tfrac{2}{3} + \tfrac{1}{3} f(A)$$

i.e. $f(A) < 1$ which is a contradiction.

If $n = 2$, then $f\left(\frac{m}{2}A+C\right) < 1$. This is clearly not possible if m is an even integer, for then $\frac{m}{2}A+C$ is a non-zero lattice point. If m is an odd integer $2k-1$ say, then we have

$$f(kA+C-\tfrac{1}{2}A) < 1.$$

Also we have $f\left(\frac{A}{2}\right) < 1$ and $f\left(-\frac{A}{2}\right) < 1$. Hence $A/2$ belongs to the three sets K, $K+A$ and $K+kA+C$, a contradiction. Thus we must have $n = 1$ and hence A, B generate Λ.

Both $B+A$ and $B-A$ cannot lie in $2K$ for then $\frac{1}{2}(A+B)$ belongs to the three sets K, $K+A$ and $K+B$. Replacing A by $-A$ if necessary we can suppose that $f(B+A) \geqslant 2$. Also we cannot have $f(B-2A) < 2$, for then $f\left(\frac{B}{2}-A\right) < 1$ which implies that $B/2$ lies in the three sets K, $K+B$ and $K+A$.

Thus we have a basis A, B of Λ satisfying

(i) $f(A) \leqslant f(C)$ for $C \in \Lambda$, $C \neq O$.

(ii) $1 \leqslant f(A) \leqslant f(B) < 2$.

(iii) $f(B+A) \geqslant 2$, $f(B-2A) \geqslant 2$.

Since $f(A) < 2$, the sets K and $K+A$ overlap. Let P be a point common to the boundaries of K and $K+A$ on the same side of OA as B. Then $P-A$ is common to the boundaries of $K-A$ and K. There are points arbitrarily near P which are already double covered by K and $K+A$. Since K is open, P cannot belong to any $K+C$ for $C \in \Lambda$. This implies that $K+P$ does not contain any lattice point. Similarly $K+P-A$ is free of lattice points. The points O, A, $2P$, $2P-A$ lie on the boundary of $K+P$ and the points O, $-A$, $2P-A$, $2P-2A$ lie on the boundary of $K+P-A$. It is then clear that the set $\bigcup_{m=-\infty}^{\infty} (K+P+mA)$ covers the entire strip between OA and the line l through $2P$ parallel to OA. Hence the point B lies on or above the line l.

The line l meets the boundary of $2K$ in the points $2P$ and $2P-2A$, so that it meets $2K$ in a segment of length $|2A|$. Since B is above this line and $2K$ is convex, the line through B parallel to OA cuts $2K$ in a segment CD of length $\leqslant |2A|$.

Let Λ' be the lattice generated by $2A$ and C. Then $d(\Lambda') = 2d(\Lambda)$. So it suffices to prove that $d(\Lambda') > 4\Delta(K)$. This follows from the following lemma:

Lemma—The lattice Λ' generated by $2A$ and C is admissible for $2K$.

Proof: Let if possible $2mA+nC$ be a point of Λ' other than 0 in $2K$. Without loss of generality we can suppose $n \geqslant 0$. Since $2mA \notin 2K$ for $|m| \geqslant 1$, we cannot have $n = 0$. If $n = 1$, then $2mA+C \notin 2K$ for this point does not belong to the open segment CD in which the line through C parallel to OA meets $2K$.

If $n \geqslant 2$, then $2mA+nC \in 2K$ implies

$$f\left(\frac{2m}{n}A+C\right) < 1.$$

Let $C = B+rA$, so that $f\left(\left(\frac{2m}{n}+r\right)A+B\right) < 1.$

Hence we shall arrive at a contradiction if we can prove that $f(B+\lambda A) \geqslant 1$ for all real λ.

Suppose $f(B+\lambda A) < 1$ for some real λ. Then clearly we must have $-2 < \lambda < 1$, since $f(B-2A) \geqslant 2, f(B) < 2$ and $f(B+A) \geqslant 2$. Also λ is not an integer for then $B+\lambda A$ is a non-zero point of Λ, so that $f(B+\lambda A) \geqslant 1$. We have three cases:

(i) $0 < \lambda < 1$.

(ii) $-2 < \lambda < -1$.

(iii) $-1 < \lambda < 0$.

We now show that in all cases we get a contradiction.

Case (i): $0 < \lambda < 1$

Since $f(B+\lambda A) < 1$, we have

$$\begin{aligned} f(A) \leqslant f(B) &= f(B+\lambda A-\lambda A) \\ &\leqslant f(B+\lambda A)+\lambda f(A) < 1+\lambda f(A) \end{aligned}$$

i.e.

$$(1-\lambda)f(A) < 1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

Also

$$\begin{aligned} 2 \leqslant f(B+A) &= f(B+\lambda A+(1-\lambda)A) \\ &\leqslant f(B+\lambda A)+(1-\lambda)f(A) \\ &< 1+(1-\lambda)f(A) \end{aligned}$$

i.e. $(1-\lambda)f(A) > 1$, which contradicts (2).

Case (ii): $-2 < \lambda < -1$

In this case we have

$$\begin{aligned} f(A) \leqslant f(B-A) &= f(B+\lambda A-(1+\lambda)A) \\ &\leqslant f(B+\lambda A)-(1+\lambda)f(A) \qquad (\text{since } 1+\lambda < 0) \end{aligned}$$

or

$$(2+\lambda)f(A) < 1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3)$$

Also,

$$\begin{aligned} 2 \leqslant f(B-2A) &= f(B+\lambda A-(2+\lambda)A) \\ &\leqslant f(B+\lambda A+(2+\lambda)f(A) \\ &< 1+(2+\lambda)f(A) \end{aligned}$$

i.e. $(2+\lambda)f(A) > 1$, which contradicts (3).

Case (iii): $-1 < \lambda < 0$

In this case we have

$$\begin{aligned} f(A) \leqslant f(B) &= f(B+\lambda A-\lambda A) \\ &\leqslant f(B+\lambda A)-\lambda f(A) \qquad \text{(since } \lambda < 0) \\ &< 1-\lambda f(A) \end{aligned}$$

i.e. $(1+\lambda)f(A) < 1$, or

$$f(A+\lambda A) < 1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Also,

$$\begin{aligned} f(A) \leqslant f(B-A) &\leqslant f(B+\lambda A-(1+\lambda)A) \\ &\leqslant f(B+\lambda A)+(1+\lambda)f(A) < 1+(1+\lambda)f(A) \end{aligned}$$

i.e. $-\lambda f(A) < 1$, or since $\lambda < 0$,

$$f(-\lambda A) < 1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Also,

$$f(B+\lambda A) < 1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The inequalities (4), (5) and (6) imply that the point $-\lambda A$ belongs to the three sets $K+A$, K and $K+B$, a contradiction.

Thus in all cases we arrive at a contradiction and the Lemma follows.

This completes the proof of Theorem 1.

§ **3.** In this section we give an example of a symmetric star domain S for which $\delta_2(s) > 2\delta(s)$.

Let

$$P_1 = (3, 0), \quad P_2 = (15, 3), \quad P_3 = (15, 15), \quad P_4 = (3, 15)$$

and

$$P_5 = (0, 3).$$

Let S be the region bounded by the line segments $P_1 P_2$, $P_2 P_3$, $P_3 P_4$, $P_4 P_5$ and their reflections in the axes and the origin (*see* Fig. 1). Then S is a symmetrical star domain with centre O. Let $a(S)$ denote its area. Then we prove the following Theorem:

Theorem 2—Let S be the star domain described above. Then

$$\delta(S) = \frac{a(S)}{900}$$

and

$$\delta_2(S) \geqslant \frac{a(S)}{443} > 2\delta(S).$$

PROOF: Let $D(S)$ denote the difference set of S. As remarked in § 2, we have

$$\delta(S) = \frac{a(S)}{\Delta(D(S))}$$

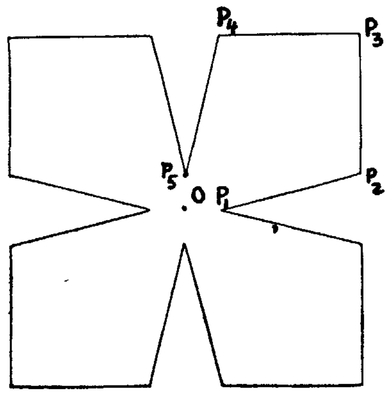

FIG. 1

where $\Delta(D(S))$ denotes the critical determinant of $D(S)$. Thus the first assertion will follow if we can show that $D(S)$ is the square

$$T: \max(|x|, |y|) \leqslant 30$$

for $\Delta(T) = \frac{1}{4}$ area of $T = 900$.

Since S is a star domain, so is $D(S)$. Also $D(S) \subset T$, since $S \subset \frac{1}{2}T$. The assertion will follow if we can prove that every point of the boundary of T is in $D(S)$. Due to symmetries of S it is enough to prove that the points $(30, \lambda) \in D(S)$ for $0 \leqslant \lambda \leqslant 30$. Since S is symmetrical about the origin, $2S \subset D(S)$. Therefore $(30, \lambda) \in D(S)$ for $6 \leqslant \lambda \leqslant 30$. For $0 \leqslant \lambda \leqslant 6$, we have

$$(30, \lambda) = (15, 9+\lambda)-(-15, 9) \in D(S).$$

Hence $D(S) = T$.

To prove the second assertion we claim that the lattice Λ generated by $A = (15, 7)$, $B = (-14, 23)$ is a double packing lattice for S. For this it suffices to prove that no point of the triangle OAB is covered more than twice by the translates of S through points of Λ. It is easy to verify that the only sets which intersect OAB are the translates of S through the points O, A, B, $-A$ and $A+B$. In Fig. 2, the parts of the triangle OAB which are covered exactly twice are shaded (by means of lines) and the portion which is single covered is dotted and the remaining portion is not covered at all. Hence Λ is a double packing lattice for S, so that

$$\delta_2(S) \geqslant \frac{a(S)}{d(\Lambda)} = \frac{a(S)}{443} > 2\delta(S).$$

This proves Theorem 2.

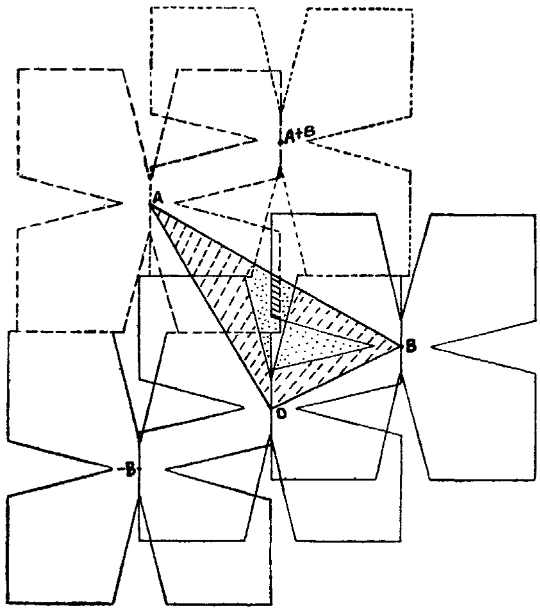

FIG. 2

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Professor L. Fejes Tóth asked whether the result of Theorem 1 is true for non-symmetrical convex domain. The answer to the question is in the negative as can be easily seen in the case of a triangle K. For $\delta(K) = 2/3$ and it is easy to verify that a lattice Λ which provides the best lattice covering for K in fact provides a lattice double packing for K also, so that $\delta_2(K) \geqslant 3/2 > 2\delta(K)$.