TEMPERATURE DISTRIBUTION OVER AN INFINITE PLATE

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In the present paper a problem of an infinite plate with heat source dependent on temperature is discussed. In the solution of the problem, Fourier number has been introduced, which has the meaning of generalized time.

Nomenclature

$\theta = \frac{T}{T_m}$ = dimensionless temperature

T_m = temperature of the medium

a = thermal diffusivity

t = time

$F_0 = \frac{at}{R^2}$ = Fourier number

$\lambda = ac\gamma$; c = specific heat

γ = density

$$\text{erfc}\, x = 1-\text{erf}\, x = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-x^2)\, dx = \text{Gauss error function}$$

$$P_0(X, F_0) = \frac{R^2}{\lambda T_m} W(r, t) = \text{Pomerantesev criterion.}$$

INTRODUCTION

Galonen (1964) studied the non-linear problem of a plane parallel wall with a heat source dependent on the temperature. Perelman (1960) considered the temperature field in a hollow finite cylinder with internal sources, which are either the function of time or space.

In the present paper, an attempt has been made to discuss a problem of an infinite plate with heat source dependent on the temperature. In the solution of the problem, Fourier number has been introduced, which has the meaning of generalized time.

STATEMENT OF THE PROBLEM

We consider an infinite plate of thickness $2R$. The initial temperature distribution is taken as $T(x, 0) = T_0$. At the initial time instant, the plate is placed into a medium of temperature $T_m > T_0$. Inside the plate, there is a heat source of specific strength W, which varies directly as

$$\frac{d^2T^2}{dx^2}.$$

The temperature distribution across the plate thickness at any time is to be found.

Mathematical Formulation

The equation of the heat with continuous heat source is

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}+\frac{W(x,t)}{c\gamma}. \quad \dots \quad (1)$$

On introduction of dimensionless quantities

$$\theta = \frac{T}{T_m}, \quad \bar{x} = \frac{x}{R}, \quad F_0 \text{ (generalized time)},$$

we obtain (1) in the form

$$\frac{\partial \theta}{\partial F_0} = \frac{\partial^2 \theta}{\partial \bar{x}^2}+P_0(\bar{x}, F_0).$$

On taking $P_0(x, F_0) = \mu \frac{d^2\theta^2}{d\bar{x}^2}$, this relation reduces to

$$\frac{\partial \theta}{\partial F_0} = \frac{\partial^2 \theta}{\partial \bar{x}^2}+\mu\frac{d^2\theta^2}{d\bar{x}^2}, \text{ where } \mu \text{ is a constant} \quad \dots \quad (2)$$

subject to the conditions

(i) $\theta(\bar{x}, 0) = \theta_0 \quad (0 < \bar{x} < \infty)$

(ii) $\theta(0, F_0) = 0$

(iii) $\frac{\partial \theta}{\partial \bar{x}}(\infty, F_0) = 0.$

Solution of the Problem

To linearize eqn. (2), Luikov (1968) suggested that, one should choose for the term of the equation containing μ, a value of the temperature which would conform to the appropriate expression if $\mu = 0$. With $\mu = 0$, we have the solution of (2) as

$$\theta = \theta_0\left[1-\operatorname{erfc}\frac{\bar{x}}{2F_0^{\frac{1}{2}}}\right].$$

Substituting the value of θ in the right-hand side of eqn. (2), we obtain

$$\frac{\partial \theta}{\partial F_0} = \frac{\partial^2 \theta}{\partial \bar{x}^2}+\mu\theta_0^2\frac{\partial^2}{\partial \bar{x}^2}\left[\operatorname{erf}\frac{\bar{x}}{2F_0^{\frac{1}{2}}}\right]^2. \quad \dots \quad (3)$$

For simplicity, we introduce a new variable $\xi = \frac{\bar{x}}{2F_0^{\frac{1}{2}}}$ in eqn. (3) and obtain

$$\frac{d^2\theta}{d\xi^2}+2\xi\frac{d\theta}{d\xi} = -\mu\theta_0^2\frac{d^2}{d^2\xi}[\text{erf }\xi]^2 \qquad (4)$$

with the conditions

$$\theta = 0 \text{ at } \xi = 0 \text{ and } \theta = \theta_0 \text{ at } \xi = +\infty. \qquad (5)$$

The solution of the above differential equation (4) can be written as

$$\theta = \phi_1(\xi)+\phi_2(\xi) \text{ erfc } \xi. \qquad (6)$$

It is easy to see that the functions ϕ_1 and ϕ_2 are determined by the equations

$$\frac{d\phi_1}{d\xi} = -\text{erfc }\xi\frac{d\phi_2}{d\xi} \qquad (7)$$

and

$$\frac{d\phi_2}{d\xi}\frac{d}{d\xi}[\text{erfc }\xi] = -\mu\theta_0^2\frac{d^2}{d\xi^2}[1-\text{erfc }\xi]^2. \qquad (8)$$

Thus, the solution of (7) and (8) yields

$$\begin{aligned}\phi_1(\xi) = {} & A+2\mu\theta_0^2\xi^2 \text{ erfc } \xi+\mu\theta_0^2(1-\text{erfc }\xi)\\ & -\frac{2\mu\theta_0^2}{\sqrt{\pi}}\xi \exp(-\xi^2)-2\mu\theta_0^2\xi(\text{erfc }\xi)^2\\ & -\mu\theta_0^2[1-(\text{erfc }\xi)^2]+\frac{4\mu\theta_0^2 \text{ erfc }\xi}{\sqrt{\pi}}\xi\bar{e}^{\xi^2}\\ & +\frac{2\mu\theta_0^2}{\pi}[1-e^{-2\xi^2}]+\mu\theta_0^2(\text{erfc }\xi)^2 \qquad (9)\end{aligned}$$

and

$$\begin{aligned}\phi_2(\xi) = {} & B-2\mu\theta_0^2\xi^2+2\mu\theta_0^2\xi^2 \text{ erfc }\xi\\ & +\mu\theta_0^2[1-\text{erfc }\xi]-\frac{2\mu\theta_0^2}{\sqrt{\pi}}\xi\bar{e}^{\xi^2}\\ & -2\mu\theta_0^2(\text{erfc }\xi)^2. \qquad (10)\end{aligned}$$

Hence

$$\begin{aligned}\theta = {} & A+B \text{ erfc }\xi-\frac{2\mu\theta_0^2}{\sqrt{\pi}}[1-\text{erfc }\xi]\xi\bar{e}^{\xi^2}\\ & +\frac{2\mu\theta_0^2}{\pi}[1-e^{-2\xi^2}]-\mu\theta_0^2(\text{erfc }\xi)^2, \qquad (11)\end{aligned}$$

where A and B are constants. To know A and B, we take help of boundary conditions (5) and after a little simplification, we obtain

$$A = \theta_0-\frac{2\mu\theta_0^2}{\pi}$$

and

$$B = \mu\theta_0^2+\frac{2\mu\theta_0^2}{\pi}-\theta_0.$$

Now, eqn. (11) can be put as

$$\theta = \theta_0 + \mu\theta_0^2 \operatorname{erfc} \frac{\bar{x}}{2F_0^{\frac{1}{2}}} + \frac{2\mu\theta_0^2}{\pi} \operatorname{erfc} \frac{\bar{x}}{2F_0^{\frac{1}{2}}}$$

$$-\theta_0 \operatorname{erfc} \frac{\bar{x}}{2F_0^{\frac{1}{2}}} - \frac{\mu\theta_0^2\bar{x}}{\sqrt{\pi F_0}} \exp\left(-\frac{\bar{x}^2}{4F_0}\right)$$

$$+\frac{\mu\theta_0^2\bar{x}}{\sqrt{\pi F_0}} \operatorname{erfc} \frac{\bar{x}}{2\sqrt{F_0}} \exp\left(-\frac{\bar{x}^2}{4F_0}\right)$$

$$-\frac{2\mu\theta_0^2}{\pi} \exp\left(-\frac{\bar{x}^2}{2F_0}\right) - \mu\theta_0^2\left(\operatorname{erfc} \frac{\bar{x}}{2\sqrt{F_0}}\right)^2. \quad \ldots \quad \ldots \quad (12)$$

The expression for temperature distribution at the surface of the plate can be obtained on putting $\bar{x} = 1$ in the above equation, viz.

$$\theta = \theta_0 + \mu\theta_0^2 \operatorname{erfc} \frac{1}{2\sqrt{F_0}} \left\{ \frac{\pi\sqrt{F_0} + 2\sqrt{F_0} + \sqrt{\pi} \exp\left(-\frac{1}{4F_0}\right)}{\pi\sqrt{F_0}} \right.$$

$$\left. -\operatorname{erfc} \frac{1}{2\sqrt{F_0}} - \frac{1}{\mu\theta_0} \right\} - \frac{\mu\theta_0^2}{\pi\sqrt{F_0}} \left\{ \sqrt{\pi} \exp\left(-\frac{1}{4F_0}\right) + 2\sqrt{F_0} \exp\left(-\frac{1}{2F_0}\right) \right\}. \quad \ldots \quad (13)$$

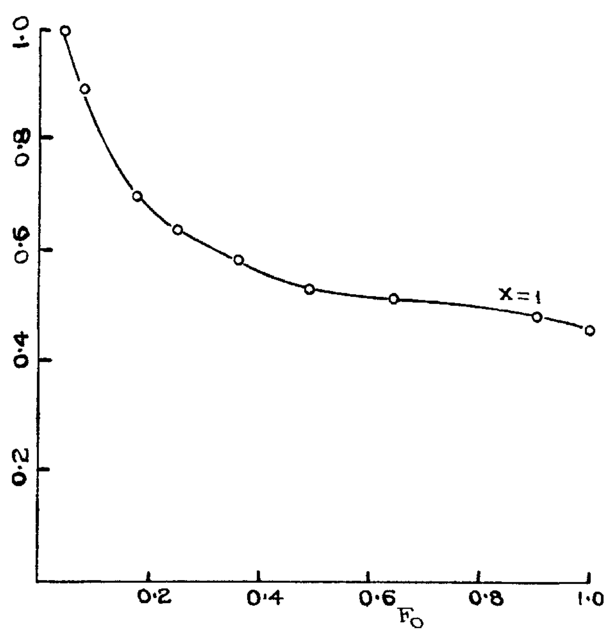

FIG. 1. Relation between θ and F_0 at $x = 1$ for plate.

To know the relation between θ and F_0, we take for simplicity $\mu = 1$ and $\theta_0 = 1$. The variation of non-dimensional temperature θ with various values of F_0 has been shown in Fig. 1. It is observed that the temperature at the surface of the plate decreases as Fourier number (generalized time) increases.

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