ON FOX'S H-FUNCTION

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Integral relation is obtained for Fox's H-function and the relation has been used to evaluate double integrals involving Bessel functions and H-functions as an application. It has been shown that how the double integrals could be evaluated easily from a known form of results or otherwise.

The object of the present paper is to study the following integral relation of Fox's H-function:

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{1-\alpha-\beta} H_{p,q}^{m,n}\left[\frac{\nu(x^2+y^2)^{h+1}}{y^{2h}} \middle| \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] f(x^2+y^2)\, dy\, dx$$

$$= \frac{\Gamma(\beta)}{4} \int_0^\infty H_{p+1,q+1}^{m+1,n}\left[\nu Z \middle| \begin{matrix}(a_p, e_p), & (\alpha+\beta, h)\\(\alpha, h) & , (b_q, f_q)\end{matrix}\right] f(Z)\, dZ \qquad \ldots \quad (1)$$

where h is a positive number and

$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0,\ \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv k > 0,\ |\arg \nu| < \tfrac{1}{2}k\pi,$ $R(\beta) > 0$; $f(Z) = O(Z^{-\delta})$ for large Z and $f(Z) = O(Z^{\epsilon-\frac{1}{2}})$ for small Z; $\delta > 0$; $\epsilon > 0$.

The H-function introduced by Fox (1961, p. 408) is represented and defined as follows:

$$H_{p,q}^{m,n}\left[Z \middle| \begin{matrix}(a_1, e_1), \ldots, (a_p, e_p)\\(b_1, f_1), \ldots, (b_q, f_q)\end{matrix}\right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1-a_j+e_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)} Z^s\, ds \qquad \ldots \quad (2)$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; e's and f's are all positive numbers, L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j - f_j s)$, $j = 1, \ldots, m$ lie on the right-hand side of the contour and those of $\Gamma(1-a_j+e_j s)$, $j = 1, 2, \ldots, n$ lie on the left-hand side of the contour.

In what follows for the sake of brevity (a_p, e_p) denotes $(a_1, e_1), \ldots, (a_p, e_p)$. Asymptotic expansion and analytic continuation of the H-function have been discussed by Braaksma (1963). Then, as an immediate consequence of the well-known formula

$$\int_0^{\pi/2} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \, d\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}, \quad R(\alpha) > 0, R(\beta) > 0, \qquad .. \quad (3)$$

we obtain our result in the form

$$\int_0^{\pi/2} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \, H_{p,q}^{m,n}\left[\nu Z(\sin\theta)^{-2h} \middle| \begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right] d\theta$$
$$= \frac{\Gamma(\beta)}{2} H_{p+1,q+1}^{m+1,n}\left[\nu Z \middle| \begin{matrix}(a_p, e_p), (\alpha+\beta, h)\\ (\alpha, h) \quad , (b_q, f_q)\end{matrix}\right] \qquad .. \quad .. \quad .. \quad .. \quad (4)$$

provided $R(\alpha) > 0$, $R(\beta) > 0$, h is a positive number and

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0, \quad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv k > 0, \quad |\arg Z| < \tfrac{1}{2}k\pi.$$

To prove (4), we substitute the contour integral (2) for the H-function and change the order of integration which is permissible due to the absolute convergence of the integrals involved in the process; then evaluating the inner integral with the help of (3) and using the definition (2) of H-function, we get the required result (4).

On putting $Z = r^2$ in (4) and then multiplying both sides by $rf(r^2)$ and integrating between the limits $(0, \infty)$ with respect to r; we have

$$\int_0^\infty rf(r^2)\, dr \int_0^{\pi/2} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \, H_{p,q}^{m,n}\left[\nu r^2(\sin\theta)^{-2h} \middle| \begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right] d\theta$$
$$= \frac{\Gamma(\beta)}{2} \int_0^\infty rf(r^2) H_{p+1,q+1}^{m+1,n}\left[\nu r^2 \middle| \begin{matrix}(a_p, e_p), (\alpha+\beta, h)\\ (\alpha, h) \quad , (b_q, f_q)\end{matrix}\right] dr. \qquad .. \quad .. \quad (5)$$

Now on putting $x = r\cos\theta$, $y = r\sin\theta$ and after some simplification, we get the required result (1).

For applications, it is shown that the double integrals can be evaluated easily by choosing $f(Z)$ in convenient form. That means, either the right-hand side integral of (1) is known after choosing $f(Z)$ (Bajpai 1969, Dahiya *in press*) or can be evaluated.

Suppose $f(Z) = Z^{\frac{1}{2h}(1-u)-1} J_w\left(Z^{\frac{1}{2h}}\right) J_v\left(Z^{\frac{1}{2h}}\right)$.

Therefore, we have from relation (1)

$$\int_0^\infty \int_0^\infty x^{2\beta-1} y^{2\alpha-1} (x^2+y^2)^{\frac{1}{2h}(1-u)-\alpha-\beta} J_w\left[(x^2+y^2)^{\frac{1}{(2h)}}\right] J_v\left[[(x^2+y^2)^{\frac{1}{(2h)}}\right]$$
$$\times H_{p,q}^{m,n}\left[\frac{\nu(x^2+y^2)^{h+1}}{y^{2h}} \middle| \begin{matrix}(a_p, e_p)\\ (b_q, f_q)\end{matrix}\right] dy\, dx$$
$$= \frac{\Gamma(\beta)}{4} \int_0^\infty Z^{\frac{1}{2h}(1-u)-1} J_w(Z^{1/2h}) J_v(Z^{1/2h}) H_{p+1,q+1}^{m+1,n}\left[\nu Z \middle| \begin{matrix}(a_p, e_p), (\alpha+\beta, h)\\ (\alpha, h) \quad , (b_q, f_q)\end{matrix}\right] dZ. \quad (6)$$

By evaluating the integral on the right side with the help of the result given by Bajpai (1969, p. 683), we get

$$\int_0^\infty \int_0^\infty x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{1}{2h}(1-u)-\alpha-\beta} J_w\left[(x^2+y^2)^{\frac{1}{(2h)}}\right] J_v\left[(x^2+y^2)^{\frac{1}{(2h)}}\right]$$
$$\times H_{p,q}^{m,n}\left[\frac{\nu(x^2+y^2)^{h+1}}{y^{2h}}\middle|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] dy\, dx$$
$$= 2^{2h-u-2}\Gamma(\beta) H_{p+5,\,q+2}^{m+2,\,n+1}\left[2^{2h}\,.\,\nu\middle|\begin{matrix}\left(\frac{1+u-w-v}{2}, h\right), (a_p, e_p), (\alpha+\beta, h),\\(u, 2h), (\alpha, h), (b_q, f_q)\end{matrix}\right.$$
$$\left.\left(\frac{u+w+v+1}{2}, h\right), \left(\frac{u-w+v+1}{2}, h\right), \left(\frac{u+w-v+1}{2}, h\right)\right] \quad .. \quad (7)$$

where h is a positive number and

$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0$, $\sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{n=m+1}^{q} f_j \equiv k > 0$, $|\arg \nu| < \frac{1}{2}k\pi$,
$R(w+v-u+2hb_i/f_i) > -1$ $(i = 1, 2, \ldots, m)$, $R(2ha_j/e_j-u) < 2h$ $(j = 1, 2, \ldots, n)$.

Similarly, by considering $f(Z) = Z^{\frac{\lambda+1}{\sigma}-1}{}_2F_1\left(\begin{matrix}a, b\\c\end{matrix}; -\delta Z^{1/\sigma}\right)$, we get

$$\int_0^\infty \int_0^\infty x^{2\beta-1}y^{2\alpha-1}(x^2+y^2)^{\frac{\lambda+1}{\sigma}-\alpha-\beta}\,.\,{}_2F_1\left(\begin{matrix}a, b\\c\end{matrix}; -\delta(x^2+y^2)^{1/\sigma}\right)$$
$$.\,H_{p,q}^{m,n}\left[\frac{\nu(x^2+y^2)^{h+1}}{y^{2h}}\middle|\begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] dy\, dx$$
$$= \frac{\sigma}{4}\frac{\Gamma(\beta)\Gamma(c)}{\Gamma(a)\Gamma(b)}\delta^{-\lambda-1} H_{p+3,\,q+3}^{m+3,\,n+1}\left[\frac{\nu}{\delta^\sigma}\middle|\begin{matrix}(-\lambda, \sigma), (a_p, e_p), (\alpha+\beta, h), (c-\lambda-1, \sigma)\\(a-\lambda-1, \sigma), (b-\lambda-1, \sigma), (\alpha, h), (b_q, f_q)\end{matrix}\right] \quad (8)$$

valid for $\sigma > 0$, $R\left(\lambda+1+\sigma \min \frac{b_i}{f_i}\right) > 0$ $(i = 1, 2, \ldots, m)$, $|\arg \delta| < \pi$,

$$R\left(\lambda+1+\sigma \max \frac{a_j-1}{e_j} - \min(a, b)\right) < 0 \; (j = 1, 2, \ldots, n)$$

including the condition of (1).

References

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