ON SOME FOUR SERIES EQUATIONS

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Formal solution of certain four series equations involving Laguerre polynomials and generalized Bateman k-functions has been obtained. The main result of this paper is that the solution of each set of four series equations can be reduced to that of solving Fredholm integral equation of second kind.

1. Introduction

Dual, triple and higher order series equations arise in the analysis of mixed boundary value problems of mathematical physics and various attempts to solve the dual and triple series equations by different methods have been made by various authors. Here we will discuss the solution of four series equations of the type:

$$\sum_{n=0}^{\infty} \frac{A_n}{(n+\alpha+1)!} L_n^{\alpha}(x) = \begin{cases} 0 \ (0 < x < a) & \quad (1.1) \\ 0 \ (b < x < c) & \quad (1.2) \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{(n+\alpha+\frac{3}{2})!} L_n^{\alpha}(x) = \begin{cases} f(x) \ (a < x < b) & \quad (1.3) \\ g(x) \ (c < x \leqslant \infty) & \quad (1.4) \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{(n+l+1)!} K_{2n}^{2l}(x) = \begin{cases} 0 \ (0 \leqslant x < a) & \quad (1.5) \\ 0 \ (b < x < c) & \quad (1.6) \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{(n+l+\frac{3}{2})!} K_{2n}^{2l}(x) = \begin{cases} f(x) \ (a < x < b) & \quad (1.7) \\ g(x) \ (c < x \leqslant \infty). & \quad (1.8) \end{cases}$$

Here $L_n^{\alpha}(x)$ is a Laguerre polynomial and $K_{2n}^{2l}(x)$ is a generalized Bateman k-function. In the above equations $f(x)$ and $g(x)$ are prescribed functions. A_n's are the unknown constants which are to be evaluated.

These sets of four series equations are extension of triple series equations. The method used here is similar to the one used by Cook (1963) to solve triple integral equations involving Bessel functions. The main result of this paper is that the solution of each set of four series equations can be reduced to that of solving Fredholm integral equation of second kind.

The analysis is purely formal and justification for inversion of order of integrations and summation have not been given while using them.

Section 5 deals with the problem of determining the coefficients A_n for four series equations involving series of Laguerre polynomials while § 6 for generalized Bateman k-functions where only the main results are given. These results can be derived by the method given in § 5.

2. Some Results and Notations

Following are the results which will be used in our further discussion. They are either known or can be derived easily from the more general results (Erdélyi 1954). The orthogonal property of Laguerre polynomials and generalized Bateman k-functions are

$$\int_0^\infty x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x)\, dx = \frac{(n+\alpha+1)!}{(n+1)!}\,\delta_{m,n} \qquad (2.1)$$

$$\int_0^\infty x^{-2l-1} K_{2n}^{2l}(x) K_{2m}^{2l}(x)\, dx = \frac{2^{2l}(n-l)!}{(n+l+1)!}\,\delta_{m,n} \qquad (2.2)$$

where $\delta_{m,n}$ is the Kronecker delta. We also have

$$\int_0^x (x-y)^{-1/2} y^\alpha L_n^\alpha(y)\, dy = \frac{(\frac{1}{2})!\ (n+\alpha+1)!}{(n+\alpha+\frac{3}{2})!}\, x^{\alpha+1/2} L_n^{\alpha+1/2}(x) \qquad (2.3)$$

$$\int_x^\infty (y-x)^{-1/2} e^{-y} L_n^{\alpha+1/2}(y)\, dy = (\tfrac{1}{2})!\ e^{-x} L_n^\alpha(x) \qquad (2.4)$$

$$\int_0^x (x-y)^{-1/2} e^{y} K_{2n}^{2l}(y)\, dy = \frac{(\frac{1}{2})!}{2^{1/2}}\, e^{x} K_{2n+1/2}^{2l+1/2}(x) \qquad (2.5)$$

$$\int_x^\infty (y-x)^{-1/2} e^{-y} y^{-l-5/4} K_{2n+1/2}^{2l+1/2}(y)\, dy = \frac{(\frac{1}{2})!\ (n+l+1)!}{2^{1/4}(n+l+\frac{3}{2})!} \times x^{-l-1} e^{-x} K_{2n}^{2l}(x). \qquad (2.6)$$

Further we shall use the notations

$$E(t) = t^{-\alpha-1/2} e^{-t},\quad E_1(t) = e^{-2t} t^{-l-5/4},$$

$$K_a(u, x) = \int_a^\infty \frac{E(t)\, dt}{\{(t-u)(t-x)\}^{1/2}}.$$

3. Variants of Abel's Integral Equations

From well-known solutions of Abel's integral equations (Sneddon 1966) we deduce the following results.

If $f(x)$ and $f'(x)$ are continuous functions in $(a \leqslant x \leqslant b)$, then the integral equations

$$f(x) = \int_a^x (x-t)^{-1/2} g(t)\, dt \qquad (3.1)$$

and

$$f(x) = \int_x^b (t-x)^{-1/2} g(t)\, dt \qquad (3.2)$$

have the solutions

$$g(t) = \frac{1}{\pi}\frac{d}{dt}\int_a^t (t-x)^{-1/2} f(x)\,dx \quad \ldots \quad (3.3)$$

and

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^b (x-t)^{-1/2} f(x)\,dx \quad \ldots \quad (3.4)$$

respectively.

4. Two Summation Results

As a preparation for the study of four series equations involving series of Laguerre polynomials or series of generalized Bateman k-function, we establish two summation formulae. The results in question are

$$K(u, x) = \sum_{n=0}^{\infty} \frac{(n+1)!}{(n+\alpha+\frac{3}{2})!} e^{-u} L_n^{\alpha}(x) L_n^{\alpha}(u) = \frac{e^x}{\pi}\int_{\max.(u,x)}^{\infty} \frac{E(t)\,dt}{\{(t-u)(t-x)\}^{1/2}} \quad (4.1)$$

$$K_1(u, x) = \sum_{n=0}^{\infty} \frac{[(n+l+1)!]^2}{2^{2l}(n-l)!\ (n+l+\frac{3}{2})!} u^{-l-1} e^{-u} K_{2n}^{2l}(x) K_{2m}^{2l}(u)$$

$$= 2^{3/4}\frac{x^{2l+1}e^x}{\pi}\int_{\max.(u,x)}^{\infty} \frac{E_1(t)\,dt}{\{(t-u)(t-x)\}^{1/2}}. \quad \ldots \quad (4.2)$$

These results can be proved easily. We shall discuss the method of (4.1) while (4.2) can be proved in the same way. Substituting the value of $L_n^{\alpha}(u)$ from (2.4) in (4.1), we get on changing the order of integration and summation

$$K(u, x) = \frac{1}{(\frac{1}{2})!}\int_u^{\infty} (y-u)^{-1/2} e^{-y} L(x, y)\,dy \quad \ldots \quad (4.3)$$

where

$$L(x, y) = \sum_{n=0}^{\infty} \frac{(n+1)!}{(n+\alpha+\frac{3}{2})!} L_n^{\alpha}(x) L_n^{\alpha+1/2}(y)$$

substituting the value of $L_n^{\alpha+1/2}(y)$ from (2.3) and using orthogonality relation (2.1), we get

$$L(x, y) = \frac{1}{(\frac{1}{2})!}(y-x)^{-1/2} e^x y^{-\alpha-1/2} H(y-x) \quad \ldots \quad (4.4)$$

where $H(t)$ is a Heaviside's unit function. The relation (4.4) is easily proved. On substituting the value of $L(x, y)$ from (4.4) in (4.3) we obtain the result.

5. The Solution of the Four Series Equations involving Series of Laguerre Polynomials

Here we assume that

$$\sum_{n=0}^{\infty} \frac{A_n}{(n+\alpha+1)!} L_n^{\alpha}(x) = \begin{cases} x^{-\alpha} m(x) \ldots (a < x < b) & \ldots \quad (5.1) \\ x^{-\alpha} n(x) \ldots (c < x \leqslant \infty) & \ldots \quad (5.2) \end{cases}$$

where $m(u)$ and $n(u)$ are unknown functions. These equations along with (1.1), (1.2) and the orthogonality relation (2.1) give

$$A_n = (n+1)!\left\{\int_a^b m(u)+\int_c^\infty n(u)\right\} e^{-u}L_n^\alpha(u)\,du. \quad \cdots \quad \cdots \quad (5.3)$$

Substituting the value of the coefficients A_n from (5.3) in eqns. (1.3) and (1.4) and further interchanging the order of integration and summation, we get

$$f(x) = \left\{\int_a^b m(u)+\int_c^\infty n(u)\right\} K(u, x)\,du \ldots (a < x < b) \quad \cdots \quad (5.4)$$

$$g(x) = \left\{\int_a^b m(u)+\int_c^\infty n(u)\right\} K(u, x)\,du \ldots (c < x \leqslant \infty). \quad \cdots \quad (5.5)$$

Equation (5.5) can be written as

$$\pi e^{-x}g(x) = \int_a^b m(u)K_x(u, x)\,du+\int_c^\infty n(u)\,du \int_{\max.(u,\,x)}^\infty \frac{E(y)\,dy}{\{(y-x)(y-u)\}^{1/2}}. \quad (5.6)$$

The integral involved in (5.6) can be rewritten, on noting the change of the order of integration,

$$\int_c^\infty du \int_{\max.(u,\,x)}^\infty dy = \int_x^\infty dy \int_c^y du.$$

(The integrand being understood).

Hence (5.6) can be written as

$$\int_x^\infty \frac{E(y)\,dy}{(y-x)^{1/2}} \int_c^y \frac{n(u)\,du}{(y-u)^{1/2}} = \pi e^{-x}g(x)- \int_a^b m(u)K_x(u, x)\,du.$$

On using (3.4), we get

$$E(y) \int_c^y \frac{n(u)\,du}{(y-u)^{1/2}} = -\frac{d}{dy}\int_y^\infty \frac{e^{-x}g(x)\,dx}{(x-y)^{1/2}}+\frac{1}{\pi}I \quad \cdots \quad \cdots \quad (5.7)$$

where

$$I = \frac{d}{dy}\int_y^\infty \frac{dx}{(x-y)^{1/2}}\int_a^b m(u)\,du \int_x^\infty \frac{E(t)\,dt}{\{(t-x)(t-u)\}^{1/2}}.$$

Inverting the order of integration we obtain

$$I = \int_a^b m(u)\,du\,\frac{d}{dy}\int_y^\infty \frac{E(t)\,dt}{(t-u)^{1/2}}\int_y^t \frac{dx}{\{(x-y)(t-x)\}^{1/2}}$$

$$= \int_a^b m(u)\,du\,\frac{d}{dy}\int_y^\infty \frac{E(t)\,dt}{(t-u)^{1/2}}\pi = -\pi \int_a^b \frac{m(u)\,E(y)\,du}{(y-u)^{1/2}}.$$

Since

$$\int_y^t \frac{dx}{(x-y)^{1/2}(t-x)^{1/2}} = \pi,$$

hence

$$\int_c^y \frac{n(u)\,du}{(y-u)^{1/2}} = F(y)- \int_a^b \frac{m(u)\,du}{(y-u)^{1/2}} \quad \cdots \quad \cdots \quad \cdots \quad (5.8)$$

where

$$F(y) = -\frac{1}{E(y)}\frac{d}{dy}\int_y^\infty \frac{e^{-x}g(x)\,dx}{(x-y)^{1/2}} \quad \ldots \quad (5.9)$$

$F(y)$ is a known function as $g(x)$ is prescribed.
Again using (3.3) and in R.H.S. on replacing u as t

$$n(u) = F_1(u) - \frac{1}{\pi}\int_a^b m(t)\,dt\,\frac{d}{du}\int_c^u \frac{dy}{(u-y)^{1/2}(y-t)^{1/2}}$$

$$n(u) = F_1(u) - \frac{1}{\pi}\int_a^b \frac{(c-t)^{1/2}m(t)\,dt}{(u-c)^{1/2}(u-t)} \quad \ldots \quad (5.10)$$

where

$$F_1(u) = \frac{1}{\pi}\frac{d}{du}\int_c^u \frac{F(y)\,dy}{(u-y)^{1/2}}.$$

Here we have used the result

$$\frac{d}{du}\int_c^u \frac{dy}{(u-y)^{1/2}(y-t)^{1/2}} = \frac{(c-t)^{1/2}}{(u-t)(u-c)^{1/2}}.$$

Again consider the eqn. (5.4) which can be written as

$$\pi e^{-x}f(x) = \int_a^b m(u)\,du\int_{\max.\,(u,\,x)}^\infty \frac{E(y)\,dy}{\{(y-x)(y-u)\}^{1/2}} + \int_c^\infty n(u)K_u(u,\,x)\,du. \quad (5.11)$$

The integral involved in (5.11) can be rewritten as on noting the change of order of integration

$$\int_a^b du\int_{\max.\,(x,\,u)}^\infty dy = \int_x^b dy\int_a^y du + \int_b^\infty dy\int_a^b du.$$

(The integrand being understood).
Hence eqn. (5.11) can be written as

$$\int_x^b \frac{E(y)M(y)\,dy}{(y-x)^{1/2}} = \pi e^{-x}f(x) - \int_b^\infty \frac{E(y)\,dy}{(y-x)^{1/2}}\int_a^b \frac{m(u)\,du}{(y-u)^{1/2}} - \int_b^\infty n(u)K_u(u,\,x)\,du \quad \ldots \quad (5.12)$$

where

$$M(y) = \int_a^y \frac{m(u)\,du}{(y-u)^{1/2}}.$$

From (3.3), we get

$$m(u) = \frac{1}{\pi}\frac{d}{du}\int_a^u \frac{M(y)\,dy}{(u-y)^{1/2}}. \quad \ldots \quad (5.13)$$

The eqn. (5.12) is an Abel's integral equation and its solution is

$$E(y)\,M(y) = -\frac{d}{dy}\int_y^b \frac{e^{-x}f(x)}{(x-y)^{1/2}} + \frac{1}{\pi}I_0 + \frac{1}{\pi}I_1 \quad \ldots \quad (5.14)$$

where

$$I_0 = \frac{d}{dy}\int_y^b \frac{dx}{(x-y)^{1/2}}\int_a^b m(u)\,du\,K_b(u,\,x)$$

and

$$I_1 = \frac{d}{dy}\int_y^b \frac{dx}{(x-y)^{1/2}} \int_c^\infty n(u)K_u(u, x)\, du.$$

First we consider

$$I_0 = \int_a^b m(u)\, du \int_b^\infty \frac{E(t)\, dt}{(t-u)^{1/2}} \frac{d}{dy}\int_y^b \frac{dx}{\{(x-y)(t-x)\}^{1/2}}$$

$$= \int_a^b m(u)\, du \int_b^\infty \frac{E(t)\, dt}{(t-u)^{1/2}} \frac{(t-b)^{1/2}}{(b-y)^{1/2}(t-y)}. \qquad (5.15)$$

Since

$$\frac{d}{dy}\int_y^b \frac{dx}{(x-y)^{1/2}(t-x)^{1/2}} = \frac{(t-b)^{1/2}}{(b-y)^{1/2}(t-y)}.$$

Secondly we consider

$$I_1 = \int_c^\infty n(u)\, du \int_u^\infty \frac{E(t)\, dt}{(t-u)^{1/2}} \frac{d}{dy}\int_y^b \frac{dx}{\{(x-y)(t-x)\}^{1/2}}$$

$$= \int_c^\infty n(u)\, du \int_u^\infty \frac{E(t)(t-b)^{1/2}\, dt}{(b-y)^{1/2}(t-u)^{1/2}(t-y)}.$$

On interchanging the order of integration

$$I_1 = \int_c^\infty \frac{E(t)(t-b)^{1/2}\, dt}{(b-y)^{1/2}(t-y)} \int_c^t \frac{n(u)\, du}{(t-u)^{1/2}}.$$

Introducing the value of last integral in the above equation, from (5.8); we get

$$I_1 = F_2(y) - \int_c^\infty \frac{E(t)(t-b)^{1/2}\, dt}{(b-y)^{1/2}(t-y)} \int_a^b \frac{m(u)\, du}{(t-u)^{1/2}}$$

where

$$F_2(y) = \int_c^\infty \frac{F(y)E(t)(t-b)^{1/2}\, dt}{(b-y)^{1/2}(t-y)^{1/2}}.$$

Hence we have from (5.15)

$$I_0 + I_1 = F_2(y) + \int_a^b \frac{m(u)\, du}{(b-y)^{1/2}}\left[\int_b^\infty \frac{(t-b)^{1/2}E(t)\, dt}{(t-u)^{1/2}(t-y)} - \int_c^\infty \frac{(t-b)^{1/2}E(t)\, dt}{(t-u)^{1/2}(t-y)}\right]$$

$$= F_2(y) + \int_b^c \frac{(t-b)^{1/2}E(t)\, dt}{(b-y)^{1/2}(t-y)} R$$

where

$$R = \int_a^b \frac{m(u)\, du}{(t-u)^{1/2}}.$$

From eqn. (5.13) we get

$$\pi R = \int_a^b \frac{dJ}{du} \frac{du}{(t-u)^{1/2}}$$

where

$$J = \frac{d}{du}\int_a^u \frac{M(s)\, ds}{(u-s)^{1/2}}.$$

On integrating by parts, substituting the value of J and inverting the order of integration in the last integral, we get

$$\pi R = \frac{1}{(t-b)^{1/2}} \int_a^b \frac{M(s)\, ds}{(b-s)^{1/2}} - \frac{1}{2} \int_a^b M(s)\, ds \int_s^b \frac{du}{(t-u)^{3/2}(u-s)^{1/2}}$$
$$= \frac{1}{(t-b)^{1/2}} \int_a^b \frac{M(s)}{(b-s)^{1/2}} \left[1 - \frac{b-s}{t-s}\right] ds$$
$$= \int_a^b \frac{M(s)(t-b)^{1/2}\, ds}{(b-s)^{1/2}(t-s)}.$$

Here we use the relation

$$\int_s^b \frac{du}{(t-u)^{3/2}(u-s)^{1/2}} = \frac{(b-s)^{1/2}}{(t-s)(t-b)^{1/2}}.$$

Hence

$$I_0 + I_1 = F_2(y) + \frac{1}{\pi} \int_a^b M(s) R(s, y)\, ds$$

where

$$R(s, y) = \int_b^c \frac{E(t)(t-b)\, dt}{(t-s)(t-y)\{(b-s)(b-y)\}^{1/2}}.$$

Hence

$$E(y)M(y) = -\frac{d}{dy} \int_y^b \frac{e^{-x} f(x)}{(x-y)^{1/2}} + \frac{F_2(y)}{\pi} + \frac{1}{\pi^2} \int_a^b M(s) R(s, y)\, ds$$

where $R(s, y)$ is a symmetrical kernel. This is a Fredholm integral equation of the second kind which determines the function $M(y)$; $m(u)$ is found from eqn. (5.13). Then eqn. (5.10) determines $n(u)$ and the coefficients A_n's are calculated from (5.3).

6. Solution of the Four Series Equations involving Series of Generalized Bateman K-functions

We started with the assumption that

$$\sum_{n=0}^{\infty} \frac{A_n}{(n+l+1)!} K_{2n}^{2l}(x) = \begin{cases} e^{-x} x^l h(x) \ldots (a < x < b) & \ldots (6.1) \\ e^{-x} x^l k(x) \ldots (c < x \leqslant \infty). & \ldots (6.2) \end{cases}$$

These equations together with eqns. (1.5) and (1.6) and the orthogonality relation (2.2) give

$$A_n = \frac{\{(n+l+1)!\}^2}{2^{2l}(n-l)!} \left[\int_a^b h(u) + \int_c^\infty k(u)\right] e^{-u} u^{-l-1} K_{2n}^{2l}(u)\, du \qquad \ldots (6.3)$$

where the functions $h(u)$ and $k(u)$ are determined from the integral equations

$$E_1(y)H(y) = \phi_2(y) + \frac{1}{\pi^2} \int_a^b H(s) R_1(s, y)\, dy \qquad \ldots \ldots (6.4)$$

$$h(u) = \frac{1}{\pi} \frac{d}{du} \int_a^u \frac{H(y)\, dy}{(u-y)^{1/2}} \qquad \ldots \ldots \ldots \ldots (6.5)$$

$$n(u) = \phi_1(u) - \frac{1}{\pi} \int_a^b \frac{(c-t)^{1/2} m(t)\, dt}{(u-c)^{1/2}(u-t)} \qquad \ldots \ldots \ldots (6.6)$$

where

$$\phi_1(u) = -\frac{1}{2^{3/4}\pi}\frac{d}{du}\int_c^u \frac{dy}{E_1(y)(u-y)^{1/2}}\left[\frac{d}{dy}\int_y^\infty \frac{x^{-2l-1}e^{-x}g(x)\,dx}{(x-y)^{1/2}}\right]$$

and

$$\phi_2(y) = -\frac{1}{2^{3/4}}\frac{d}{dy}\int_y^\infty \frac{x^{-2l-1}e^{-x}f(x)\,dx}{(x-y)^{1/2}} - \frac{1}{2^{3/4}\pi}\int_c^\infty \frac{E(t)(b-t)^{1/2}\,dt}{(b-y)^{1/2}(t-y)}$$
$$\times\left[\frac{d}{dy}\int_y^\infty \frac{x^{-2l-1}e^{-x}g(x)\,dx}{(x-y)^{1/2}}\right]$$

are known functions since $f(x)$ and $g(x)$ are prescribed and

$$R_1(s, y) = \int_b^c \frac{E_1(t)(b-t)\,dt}{(s-t)(y-t)\{(b-y)(s-b)\}^{1/2}}$$

is a symmetrical kernel.

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