GENERALIZED HYPERGEOMETRIC FUNCTION AND HEAT PRODUCTION IN A SEMI-INFINITE CYLINDER

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In this paper the author evaluates an integral involving the product of generalized hypergeometric function and Fox's H-function. Then a problem of heat conduction in a semi-infinite circular cylinder when there are sources of heat within the cylinder is solved. Lastly, by making use of the integral evaluated earlier, the solution of the problem by taking heat source as product of exponential function and a generalized hypergeometric function is obtained.

On account of the general nature of the boundary conditions and heat source considered, the solution of the problem of heat conduction yields many useful and interesting cases. Also the integral evaluated is quite general and is capable of giving a number of new and interesting integrals as its special cases.

1. INTRODUCTION

The solution of the heat equation

$$\frac{\partial \theta}{\partial t} = k\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right) + \phi(r, t) \qquad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

under the following boundary and initial conditions

$$\theta = 0 \text{ when } r = a,\ t > 0$$

$$\theta = 0 \text{ when } t = 0,\ 0 \leqslant r \leqslant a$$

has been recently obtained by many authors by taking the heat source $\phi(r, t)$ in terms of various special functions. Bhonsle (1966), for example, studied the use of the Jacobi polynomials; Bajpai (1968, 1969) the Meijer's G-function and the associated Legendre functions while Sharma (1969) Fox's H-function.

The aim of this paper is to solve the following problem of heat conduction in a semi-infinite circular cylinder of radius a bounded by the plane $z = 0$ and the cylindrical surface $r = a$:

$$\frac{\partial \theta}{\partial t} = k\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2}\right) + \phi(r, z, t) \qquad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

$$\theta = \theta_0, \text{ a constant},\ z = 0,\ 0 \leqslant r \leqslant a,\ t > 0 \qquad \ldots \quad \ldots \quad (1.3)$$

$$\theta = g(z, t), \qquad r = a,\ z > 0,\ t > 0 \qquad \ldots \quad \ldots \quad (1.4)$$

$$\theta(r, z, 0) = f(r, z), \qquad 0 \leqslant r \leqslant a,\ z > 0 \qquad \ldots \quad \ldots \quad (1.5)$$

the functions $g(z, t)$ and $f(r, z)$ being given.

We further suppose that

$$\left.\begin{aligned} \phi(r, z, t) &= k/K \,.\, \phi_1(r) \,.\, \phi_2(z) \,.\, \phi_3(t) \\ g(z, t) &= g_1(z) \,.\, g_2(t) \\ f(r, z) &= f_1(r) \,.\, f_2(z) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

where k is the diffusivity and K the conductivity of the material.

In this paper we consider a set of values of these functions and in particular take

$$\phi_1(r) = r^{\sigma-2}\left(1-\frac{r}{a}\right)^{\rho-1} {}_hF_s\left(\begin{matrix} c_1, \ldots, c_h \\ d_1, \ldots, d_s \end{matrix}; 1-\frac{r}{a}\right)$$

where ${}_hF_s\left(\begin{matrix} c_1, \ldots, c_h \\ d_1, \ldots, d_s \end{matrix}; x\right)$ is the generalized hypergeometric function defined as

$${}_hF_s\left(\begin{matrix} c_1, \ldots, c_h \\ d_1, \ldots, d_s \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{h} (c_j)_n}{\prod_{j=1}^{s} (d_j)_n} \cdot \frac{x^n}{n!}. \quad \ldots \quad \ldots \quad (1.7)$$

The series in (1.7) is absolutely convergent for $h < s+1$ or $h = s+1$, $|x| \leqslant 1$, $\mathrm{Re}\left(\sum_{j=1}^{s} d_j - \sum_{j=1}^{h} c_j\right) > 0$ and none of d_j is a negative integer or zero. The heat source of this character may encompass several cases of interest due to the general nature of the function ${}_hF_s\left(\begin{matrix} c_1, \ldots, c_h \\ d_1, \ldots, d_s \end{matrix}; x\right)$.

In section 2, we establish an integral involving the product of Fox's H-function and the generalized hypergeometric function. This integral is required for our subsequent work in this paper.

2. The Integral

The H-function of Fox (1961) is defined and represented as:

$$H_{p,q}^{m,n}\left[x \middle| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix}\right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+\beta_j \xi)} \cdot \frac{\prod_{j=1}^{n} \Gamma(1-a_j+\alpha_j \xi)}{\prod_{j=n+1}^{p} \Gamma(a_j-\alpha_j \xi)} \cdot x^{\xi}\, d\xi \quad \ldots \quad (2.1)$$

where an empty product is interpreted as unity; $0 \leqslant m \leqslant q$, $0 \leqslant n < p$ and α_j $(j = 1, \ldots, p)$ and β_h $(h = 1, \ldots, q)$ are positive numbers. The poles of the integrand are simple. The contour C is a straight line parallel to the imaginary axis such that the poles of $\Gamma(b_j - \beta_j \xi)$ $(j = 1, \ldots, m)$ lie to its right and those of $\Gamma(1-a_j+\alpha_j \xi)$ $(j = 1, \ldots, n)$ lie to its left. When $\alpha_i = \beta_j = 1$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$, the H-function reduces to Meijer's G-function (Erdélyi 1953).

Braaksma (1963) has obtained the conditions of convergence of the integral (2.1) and the asymptotic expansions of the H-function.

In this paper (a_p) will stand for $a_1, \ldots, a_p$; $\{(a_p, \alpha_p)\}$ for $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$; $\Delta(k, \alpha)$ for $\alpha/k, (\alpha+1)/k, \ldots, (\alpha+k-1)/k$; $\{\Delta(k, \alpha), \beta\}$ for $(\alpha/k, \beta), ((\alpha+1)/k, \beta), \ldots, ((\alpha+k-1)/k, \beta)$ and $(a)_m$ for $\Gamma(a+m)/\Gamma(a)$.

Here we establish the following integral:

$$\int_0^1 x^{\rho-1}.(1-x)^{\sigma-1}.{}_hF_s\left(\begin{matrix}c_1, \ldots, c_h\\ d_1, \ldots, d_s\end{matrix}; ax^k\right)$$

$$\times H_{p,q}^{m,n}\left[z(1-x)^\lambda \left| \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] dx$$

$$= \Gamma(\rho).\sum_{V=0}^{\infty} \frac{\prod_{j=1}^{h}(c_j)_V}{\prod_{j=1}^{s}(d_j)_V}.\frac{a^V}{V!}.\prod_{j=1}^{k}\left(\frac{\rho+j-1}{k}\right)_V$$

$$\times H_{p+k+1,\, q+k+1}^{m,\, n+k+1}\left[z \left| \begin{matrix}\left\{\Delta(k, 1-\rho-\sigma), \frac{\lambda}{k}\right\}, (1-\sigma, \lambda), \{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}, (1-\rho-\sigma, \lambda), \left\{\Delta(k, 1-\rho-\sigma-kV), \frac{\lambda}{k}\right\}\end{matrix}\right.\right] \tag{2.2}$$

where Re $(\rho) > 0$, Re $(\sigma) > 0$, k is a positive integer; $\lambda > 0$, $A \equiv \sum_{j=1}^{n}(\alpha_j) + \sum_{j=1}^{m}(\beta_j) - \sum_{j=n+1}^{p}(\alpha_j) - \sum_{j=m+1}^{q}(\beta_j) > 0$, $|\arg. z| < 1/2A\pi$ and Re $\left(\sigma+\lambda\frac{b_j}{\beta_j}\right) > 0$ $(j = 1, \ldots, m)$.

PROOF: To establish (2.2), we substitute the value of the H-function from (2.1) in the integrand of (2.2), interchange the order of integration which is justified under the conditions stated (due to the absolute convergence of the integrals involved in the process) and get

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)}.\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}.z^\xi\, d\xi$$

$$\times\left(\int_0^1 x^{\rho-1}(1-x)^{\sigma+\lambda\xi-1}.{}_hF_s\left(\begin{matrix}c_1, \ldots, c_h\\ d_1, \ldots, d_s\end{matrix}; ax^k\right)dx\right).$$

On evaluating the x-integral with the help of Rainville's (1963) result, expressing ${}_{h+k}F_{s+k}(x)$ in terms of an infinite series as defined in (1.7), interchanging the order of integration and summation (which is permissible under the stated conditions) and interpreting the result with the help of (2.1), the desired result is established.

Corollary—On taking $a = k = \alpha_i = \beta_j = 1$ $(i = 1, \ldots, p, j = 1, \ldots, q)$ in (2.2), we get the following integral:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}{}_hF_s\left(\begin{matrix} c_1, \ldots, c_h \\ d_1, \ldots, d_s \end{matrix}; x\right) G_{p,q}^{m,n}\left[z(1-x)^\lambda \middle| \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right] dx$$

$$= \sum_{V=0}^{\infty} \frac{\prod_{j=1}^{h} (c_j)_V}{\prod_{j=1}^{s} (d_j)_V} \cdot \frac{\Gamma(\rho+V)}{V!} \cdot \frac{1}{\lambda^{\rho+V}}$$

$$\times G_{p+\lambda,q+\lambda}^{m,n+\lambda}\left[z \middle| \begin{matrix} \Delta(\lambda, 1-\sigma), (a_p) \\ (b_q), \Delta(\lambda, 1-\rho-\sigma-V) \end{matrix}\right] \quad \ldots \quad \ldots \quad (2.3)$$

where Re $(\rho) > 0$, Re $(\sigma) > 0$, λ is a positive integer, Re $(\sigma+\lambda b_j) > 0$ $(j = 1, 2, \ldots, m)$, $\delta = (m+n)-1/2(p+q) > 0$, and $|\arg. z| < \pi\delta$.

It may be remarked here that in case $|\arg. z| = \delta\pi$, $\delta \geqslant 0$, the integral in (2.3) is valid if the following conditions are satisfied (Luke 1969):

(i) Re $(\rho) > 0$, Re $(\sigma) > 0$, λ is a positive integer, Re $(\sigma+\lambda b_j) > 0$, and

(ii) Re $\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\right) > 1$ when $p = q$; and when $p \neq q$, if with $\xi = \mu+i\tau$, μ and τ real, μ is chosen so that for $\sigma \to \pm\infty$,

$$(q-p)\mu > \text{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) + 1 - 1/2(q-p).$$

3. Fourier Sine and Finite Hankel Transforms

Let the Fourier sine transform of $\theta(r, z, t)$ be

$$\bar{\theta}(r, \xi, t) = \int_0^\infty \theta(r, z, t) \sin \xi z \, dz, \quad \xi > 0. \quad \ldots \quad \ldots \quad (3.1)$$

Then, we have (Erdélyi 1954*a*)

$$\theta(r, z, t) = \frac{2}{\pi} \int_0^\infty \bar{\theta}(r, \xi, t) \sin \xi z \, d\xi, \quad z > 0. \quad \ldots \quad \ldots \quad (3.2)$$

Also, let the finite Hankel transform of $u(r)$ be (Snnedon 1951)

$$J[u(r); \eta_i] = \int_0^a r J_0(\eta_i r) u(r) \, dr = \bar{\bar{u}}(\eta_i). \quad \ldots \quad \ldots \quad (3.3)$$

Then, on taking $m = 1$, $n = 0 = p$, $q = 2$, $b_1 = b_2 = 0$, $\lambda = 2$, $x = 1-r/a$ and $z = 1/4\, a^2\eta_i^2$ in (2.3) and using Erdélyi (1954*b*), we obtain:

$$J\left[r^{\sigma-2}\left(1-\frac{r}{a}\right)^{\rho-1} {}_hF_s\left(\begin{matrix}c_1, \ldots, c_h \\ d_1, \ldots, d_s\end{matrix}; 1-\frac{r}{a}\right); \eta_i\right]$$

$$= a^{\sigma} \cdot \sum_{V=0}^{\infty} \frac{\prod_{j=1}^{h} (c_j)_V}{\prod_{j=1}^{s} (d_j)_V} \cdot \frac{\Gamma(\rho+V)\Gamma(\sigma)}{\Gamma(\rho+V+\sigma)} \cdot \frac{1}{V!}$$

$$\times {}_2F_3\left(\begin{matrix}\frac{\sigma}{2}, \quad \frac{1+\sigma}{2} \\ 1, \frac{\rho+\sigma+V}{2}, \frac{1+\rho+\sigma+V}{2}\end{matrix}; -\tfrac{1}{4}a^2\eta_i^2\right) \qquad \ldots \quad \ldots \quad (3.4)$$

where Re $(\rho) > 0$, Re $(\sigma) > 0$ and η_i is a root of the transcendental equation

$$J_0(a\eta_i) = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

Because of the inversion theorem (Snnedon 1951), we have:

$$r^{\sigma-2}\left(1-\frac{r}{a}\right)^{\rho-1} {}_hF_s\left(\begin{matrix}c_1, \ldots, c_h \\ d_1, \ldots, d_s\end{matrix}; 1-\frac{r}{a}\right)$$

$$= 2a^{\sigma-2}\,\Gamma(\sigma) \cdot \sum_i \left[\sum_{V=0}^{\infty} \frac{\prod_{j=1}^{h} (c_j)_V}{\prod_{j=1}^{s} (d_j)_V} \cdot \frac{\Gamma(\rho+V)}{\Gamma(\rho+V+\sigma)} \cdot \frac{1}{V!}\right.$$

$$\left. \times {}_2F_3\left(\begin{matrix}\frac{\sigma}{2}, \quad \frac{1+\sigma}{2} \\ 1, \frac{\rho+\sigma+V}{2}, \frac{1+\rho+\sigma+V}{2}\end{matrix}; -\tfrac{1}{4}a^2\eta_i^2\right)\right] \frac{J_0(r\eta_i)}{[J_1(a\eta_i)]^2} \qquad \ldots \quad (3.6)$$

where the sum is to be taken over all the positive roots of (3.5).

(3.6) is used later in verifying the solution of the problem mentioned in (1.2) to (1.5).

4. Solution of the Problem

To solve (1.2), we first employ the Fourier sine transform $\bar{\theta}(r, \xi, t)$ defined by (3.1). Then, since $\theta = \theta_0$ when $z = 0$, we find that

$$\int_0^{\infty} \frac{\partial^2\theta}{\partial z^2} \sin \xi z\, dz = \xi\theta_0 - \xi^2\bar{\theta}(r, \xi, t)$$

provided θ and $\partial\theta/\partial z$ both tend to zero as $z \to \infty$, so that on multiplying eqn. (1.1) by $\sin \xi z$ and integrating with respect to z between limits $(0, \infty)$

we obtain the partial differential equation

$$\frac{\partial\bar{\theta}}{\partial t}=k\left(\frac{\partial^2\bar{\theta}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\theta}}{\partial r}+\xi\theta_0-\xi^2\bar{\theta}\right)+\bar{\phi}(r,\xi,t),$$

$$0\leqslant r\leqslant a,\ t>0 \qquad \ldots \qquad (4.1)$$

for the function $\bar{\theta}(r, \xi, t)$, where

$$\bar{\phi}(r,\xi,t)=\int_0^\infty \phi(r,z,t)\sin\xi z\,dz. \qquad \ldots \qquad (4.2)$$

We now employ the finite Hankel transform $\bar{\bar{\theta}}(\eta_i, \xi, t)$ of $\bar{\theta}(r, \xi, t)$ defined in (3.3), to transform eqn. (4.1) to an ordinary differential equation.

Then, since $\qquad \bar{\theta}(a,\xi,t)=\int_0^\infty g(z,t)\sin\xi z\,dz=\bar{g}(\xi,t)$, and

$$\int_0^a r\left(\frac{\partial^2\bar{\theta}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\theta}}{\partial r}\right)J_0(r\eta_i)\,dr=\eta_i a\,J_1(\eta_i a)\bar{g}(\xi,t)-\eta_i^2\bar{\bar{\theta}}$$

on multiplying both sides of eqn. (4.1) by $r\,J_0(\eta_i r)$ and integrating over r between limits $(0, a)$, we find that $\bar{\bar{\theta}}(\eta_i, \xi, t)$ satisfies the first order equation

$$\frac{d\bar{\bar{\theta}}}{dt}+k(\xi^2+\eta_i^2)\bar{\bar{\theta}}=ka\eta_i\,J_1(\eta_i a)\bar{g}(\xi,t)$$

$$+k\xi\,.\,\frac{a}{\eta_i}\,\theta_0\,J_1(\eta_i a)+\bar{\bar{\phi}}(\eta_i,\xi,t),\quad t>0 \qquad \ldots \qquad (4.3)$$

where

$$\bar{\bar{\phi}}(\eta_i,\xi,t)=\int_0^a r\,J_0(\eta_i r)\bar{\phi}(r,\xi,t)\,dr. \qquad \ldots \qquad (4.4)$$

The solution of (4.3) can easily be shown to be

$$\bar{\bar{\theta}}(\eta_i,\xi,t)=\Bigg\{ka\eta_i\,J_1(\eta_i a)\int_0^t \bar{g}(\xi,\tau)e^{-k(\xi^2+\eta_i^2)(t-\tau)}\,d\tau$$

$$+\frac{a\xi\theta_0}{\eta_i}J_1(\eta_i a)\,.\,\frac{\left(1-e^{-k(\xi^2+\eta_i^2)t}\right)}{(\xi^2+\eta_i^2)}$$

$$+\int_0^t \bar{\bar{\phi}}(\eta_i,\xi,\tau)\,.\,e^{-k(\xi^2+\eta_i^2)(t-\tau)}\,d\tau$$

$$+F(\eta_i,\xi)e^{-k(\xi^2+\eta_i^2)t}\Bigg\} \qquad \ldots \qquad (4.5)$$

since from (1.5), $\theta=f(r, z)$ when $t=0$, so that

$$(\bar{\bar{\theta}})_{t=0}=\int_0^\infty \sin\xi z\,dz\left(\int_0^a rf(r,z)J_0(\eta_i r)\,dr\right)=F(\eta_i\,\xi). \qquad \ldots \qquad (4.6)$$

Now, the solution to the boundary value problem can easily be obtained from (4.5) by first inverting it on applying Hankel inversion theorem (Snnedon 1951) and then inverting the resulting equation in $\bar{\theta}(r, \xi, t)$ by the application of Fourier inversion theorem mentioned in (3.2). The solution is found to be as given below:

$$\theta(r, z, t) = \frac{4}{a^2\pi}\sum_i \frac{J_0(\eta_i r)}{[J_1(\eta_i a)]^2}$$
$$\times\Bigg\{\frac{a\theta_0}{\eta_i}\,.\,J_1(\eta_i a)\int_0^\infty \frac{\xi}{\xi^2+\eta_i^2}(1-e^{-k(\xi^2+\eta_i^2)t})\sin\xi z\,d\xi$$
$$+ka\eta_i\,J_1(\eta_i a)\int_0^\infty \sin\xi z\,d\xi\left(\int_0^t \bar{g}(\xi,\tau)e^{-k(\xi^2+\eta_i^2)(t-\tau)}\,d\tau\right)$$
$$+\int_0^\infty \sin\xi z\,d\xi\left(\int_0^t \bar{\bar{\phi}}(\eta_i,\xi,\tau)e^{-k(\xi^2+\eta_i^2)(t-\tau)}\,d\tau\right)$$
$$+\int_0^\infty e^{-k(\xi^2+\eta_i^2)t}F(\eta_i,\xi)\sin\xi z\,d\xi\Bigg\} \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

where the sum is taken over the positive roots of the transcendental equation (3.5).

5. Particular Cases

Corollary 1—If the surfaces $z = 0$ and $r = a$ be both kept at zero temperature and there be no heat sources within the cylinder, then our solution given by (4.7) reduces to the case considered by Snnedon (1951).

Corollary 2—On taking $g(z, t) = g_1(z)\,.\,g_2(t)$, $f(r, z) = f_1(r)\,.\,f_2(z)$ and $\phi(r, z, t) = \phi_1(r)\,.\,\phi_2(z)\,.\,\phi_3(t)$ in (4.7) and making use of Erdélyi's (1954*a*) results therein, it reduces to the following form:

$$\theta(r, z, t) = \frac{4}{a^2\pi}\sum_i \frac{J_0(\eta_i r)}{[J_1(\eta_i a)]^2}$$
$$\times\Bigg\{\frac{a\theta_0}{\eta_i}J_1(\eta_i a)\left[\frac{\pi}{2}e^{-\eta_i z}-\frac{\pi}{4}e^{-\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt}-\frac{z}{2\sqrt{kt}}\right)\right.$$
$$\left.+\frac{\pi}{4}e^{\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt}+\frac{z}{2\sqrt{kt}}\right)\right]$$
$$+ka\eta_i\,J_1(\eta_i a)\int_0^\infty R(\eta_i,\xi,t)\bar{g}_1(\xi)\sin\xi z\,d\xi$$
$$+\bar{\bar{\phi}}_1(\eta_i)\,.\int_0^\infty P(\eta_i,\xi,t)\bar{\phi}_2(\xi)\sin\xi z\,d\xi$$
$$+\bar{\bar{f}}_1(\eta_i)\int_0^\infty e^{-k(\xi^2+\eta_i^2)t}\,.\bar{f}_2(\xi)\sin\xi z\,d\xi\Bigg\} \quad \ldots \quad \ldots \quad \ldots \quad (5.1)$$

where

$$R(\eta_i,\xi,t) = \int_0^t g_2(\tau)e^{-k(\xi^2+\eta_i^2)(t-\tau)}\,d\tau \quad \ldots \quad \ldots \quad (5.2)$$

$$P(\eta_i,\xi,t) = \int_0^t \phi_3(\tau)e^{-k(\xi^2+\eta_i^2)(t-\tau)}\,d\tau \quad \ldots \quad \ldots \quad (5.3)$$

and the sum is to be taken over the positive roots of (3.5).

6. Heat Source of General Character

In this section, we obtain the solution of the boundary value problem discussed in cor. 2 above by considering the following set of values of the various functions involved therein.

Let

$$\phi_1(r) = r^{\sigma-2}\left(1-\frac{r}{a}\right)^{\rho-1} {}_hF_s\left(\begin{matrix}c_1, \ldots, c_h\\ d_1, \ldots, d_s\end{matrix}; 1-\frac{r}{a}\right), \quad \sigma \geqslant 2, \rho \geqslant 1$$

$\phi_2(z) = e^{-\mu z}$, $\mu > 0$ and constant, $\phi_3(t) = \phi_0$, a constant;

$g_1(z) = ze^{-\frac{1}{2}z^2}$, $g_2(t) = g_0$, a constant; $f_1(r) = f_0$, a constant and $f_2(z) = ze^{-\frac{1}{2}z^2}$.

Substituting the values or concerned transforms of these functions in (5.1) and then evaluating the various integrals thus obtained by virtue of Erdélyi (1954*a*) and (3.4), we finally get the following solution of the boundary value problem referred above:

$$\theta(r, z, t) = \sum_i \frac{J_0(\eta_i r)}{[J_1(\eta_i a)]^2}$$

$$\times\Bigg[\frac{1}{a\eta_i}\theta_0 J_1(\eta_i a)\Big\{2e^{-\eta_i z}-e^{-\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt}-\frac{z}{2\sqrt{kt}}\right)$$

$$+e^{\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt}+\frac{z}{2\sqrt{kt}}\right)\Big\}+\sqrt{\frac{\pi}{2}}\cdot g_0\cdot\eta_i\cdot\frac{1}{a}J_1(\eta_i a)$$

$$\times\Big\{e^{\frac{1}{2}\eta_i^2-\eta_i z}\operatorname{Erfc}\left(\frac{\eta_i-z}{\sqrt{2}}\right)-e^{\frac{1}{2}\eta_i^2+\eta_i z}\operatorname{Erfc}\left(\frac{\eta_i+z}{\sqrt{2}}\right)$$

$$-e^{\frac{1}{2}\eta_i^2-\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt+\tfrac{1}{2}}-\frac{z}{2\sqrt{kt+\frac{1}{2}}}\right)+e^{\frac{1}{2}\eta_i^2+\eta_i z}\cdot\operatorname{Erfc}\left(\eta_i\sqrt{kt+\tfrac{1}{2}}+\frac{z}{2\sqrt{kt+\frac{1}{2}}}\right)\Big\}$$

$$+\frac{\phi_0 a^{\sigma-2}}{k(\eta_i^2-\mu^2)}\cdot\left(\sum_{V=0}^{\infty}\frac{\prod_{j=1}^{h}(c_j)_V}{\prod_{j=1}^{s}(d_j)_V}\cdot\frac{\Gamma(\rho+V)\Gamma(\sigma)}{\Gamma(\rho+V+\sigma)}\cdot\frac{1}{V!}\cdot{}_2F_3\left(\begin{matrix}\frac{\sigma}{2}, & \frac{1+\sigma}{2}\\ 1, \frac{\rho+\sigma+V}{2}, & \frac{1+\rho+\sigma+V}{2}\end{matrix}; -\tfrac{1}{4}a^2\eta_i^2\right)\right)$$

$$\times\Big\{2(e^{-\mu z}-e^{-\eta_i z})-e^{-kt\eta_i^2+kt\mu^2-\mu z}\operatorname{Erfc}\left(\mu\sqrt{kt}-\frac{z}{2\sqrt{kt}}\right)$$

$$+e^{-kt\eta_i^2+kt\mu^2+\mu z}\operatorname{Erfc}\left(\mu\sqrt{kt}+\frac{z}{2\sqrt{kt}}\right)$$

$$+e^{-\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt}-\frac{z}{2\sqrt{kt}}\right)-e^{\eta_i z}\operatorname{Erfc}\left(\eta_i\sqrt{kt}+\frac{z}{2\sqrt{kt}}\right)\Big\}$$

$$+\frac{2f_0 z}{a\eta_i(2kt+1)^{\frac{3}{2}}}\cdot e^{-kt\eta_i^2-\frac{z^2}{2(2kt+1)}}\Bigg] \qquad \ldots \quad (6.1)$$

where the sum is to be taken over the positive roots of (3.5) and $\rho \geqslant 1$, $\sigma \geqslant 2$.

7. Verification of the Solution (6.1)

On substituting the value of $\theta(r, z, t)$, obtained in (6.1), in (1.2) and making use of (3.6), we see that (1.2) is satisfied. Also, from (6.1), we see that when $z = 0$, $\theta = \theta_0$ and when $t = 0$, $\theta = f_0 z e^{-\frac{1}{2}z^2}$ since when $z > 0$, Erfc $\left(\frac{z}{2\sqrt{kt}}\right) \to 0$ and Erfc $\left(-\frac{z}{2\sqrt{kt}}\right) \to 2$ as $t \to 0+$ (McLachlan (1962)). The solution (6.1) is thus completely established.

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