SOME RESULTS CONNECTED WITH REFLEXIVE BANACH SPACES

by M. K. Chakraborty, *Department of Mathematics, Kalyani University, Kalyani, Nadia, West Bengal*

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Let X be a reflexive Banach space with a basis and Y a normed linear space. Let A be a continuous linear operator on X onto Y possessing inverse. It is shown that Y is also a reflexive Banach space with a basis. Also a mapping T has been established between the second normed conjugate spaces of two reflexive Banach spaces X and Y with the help of a given mapping T_1 of X into Y and certain properties of T in relation to the mapping T_1 are obtained.

1. Introduction

Let X be a Banach space, then the normed conjugate space X' is the space of all continuous linear functionals on X. It is known (Taylor 1957, p. 185) that X' is also a Banach space. We denote the normed conjugate of X' by X''. There exists a mapping J on X into X'' defined by

$$Jx(x') = x'(x), \quad \text{for } x \in X \text{ and for all } x' \in X'.$$

i.e.

$$Jx = x'', \quad \text{where } x''(x') = x'(x) \text{ for all } x' \in X'.$$

If the range of J is all of X'' then X is said to be reflexive. The sequence $\{y_i\}_{i=1}^{\infty}$, $y_i \in X$, is a basis of X if any element $x \in X$ can be expressed uniquely in the form $x = \sum_{i=1}^{\infty} a_i y_i$, a_i are scalars.

Now let X be a Banach space with a basis. The following definitions have been introduced by Singer (1961–62).

Definition 1—For any sequence $\{y_i\}_{i=1}^{\infty}$, $y_i \in X$, $[y_i]$ denotes the closed linear subspace spanned by $\{y_i\}_{i=1}^{\infty}$. A sequence $\{y_i\}_{i=1}^{\infty}$ in X is called a basic sequence if it forms a basis in $[y_i]$.

Definition 2—A basic sequence $\{x_n\}_{n=1}^{\infty}$ is called boundedly complete if for any bounded sequence $\left\{ \sum_{i=1}^{n} a_i x_i \right\}_{n=1}^{\infty}$, a_i are scalars, $\sum_{i=1}^{\infty} a_i x_i$ converges in X.

Singer (1961–62) has obtained a criterion for a reflexive Banach space with a basis in terms of basic sequences which may be stated below.

Theorem A—Let X be a Banach space with a basis. Then the following properties are equivalent:

(i) X is reflexive

(ii) Every basic sequence in X is boundedly complete.

The purpose of the present paper is to prove two theorems and certain lemmas. In Theorem 1 we show that if X and Y are two normed linear spaces and A is a mapping of X into Y and if X is a reflexive Banach space with a basis then so is Y provided the mapping A satisfies certain conditions. To prove this theorem, we have taken help of the above result of Singer. In Theorem 2 we have proved certain properties of a mapping between the second normed conjugate spaces of two reflexive Banach spaces X and Y which is established with the help of a given mapping of X into Y. The definitions and terminologies used throughout the paper are those from Taylor (1957).

2. Lemmas

Lemma 1—Let X be a Banach space and Y a normed linear space. Let A be a continuous linear operator on X onto Y possessing inverse. Then Y is a Banach space.

Proof: The proof is elementary and follows from Goldberg (1966).

Lemma 2—Under the hypotheses of Lemma 1, if X possesses a basis then the Banach space Y also possesses a basis.

Proof: Let $\{e_i\}_{i=1}^{\infty}$ be a basis in X and suppose that $A(e_i) = \eta_i$, $i = 1, 2, \ldots$. We shall show that $\{\eta_i\}_{i=1}^{\infty}$ is a basis in Y.

Let $y \in Y$. Then there exists a unique $x \in X$ such that $A(x) = y$. Let $x = \sum_{i=1}^{\infty} \xi_i e_i$, ξ_i scalars, then $S_n = \sum_{i=1}^{n} \xi_i e_i \to x$ as $n \to \infty$ in the sense that $\| S_n - x \| \to 0$ as $n \to \infty$. Continuity of A then gives that $A(S_n) \to A(x) = y$ as $n \to \infty$. Now $A(S_n) = \sum_{i=1}^{n} \xi_i \eta_i$ and so $y = \sum_{i=1}^{\infty} \xi_i \eta_i$.

For uniqueness it is required to prove that $\sum_{i=1}^{\infty} a_i \eta_i = 0$ implies $a_i = 0$, $i = 1, 2, \ldots$. Now we have

$$0 = \sum_{i=1}^{\infty} a_i \eta_i = \sum_{i=1}^{\infty} A(a_i e_i) = \operatorname*{lt}_{n \to \infty} \left\{ \sum_{i=1}^{n} A(a_i e_i) \right\}$$

$$= \operatorname*{lt}_{n \to \infty} A\left(\sum_{i=1}^{n} a_i e_i \right) = A\left(\sum_{i=1}^{\infty} a_i e_i \right).$$

Since A^{-1} exists, $\sum_{i=1}^{\infty} a_i e_i = 0$. Again since $\{e_i\}_{i=1}^{\infty}$ is a basis in X, we have $a_i = 0$, $i = 1, 2, \ldots$. This proves the uniqueness and consequently the lemma.

Lemma 3—Under the hypotheses of Lemma 1, if $\{x_n\}_{n=1}^{\infty}$ is a basic sequence in the Banach space X with a basis, then $\{y_n\}_{n=1}^{\infty}$, $y_n = A(x_n)$, is a basic sequence in Y.

PROOF: We shall have to show that $\{y_n\}_{n=1}^{\infty}$ is a basis of $[y_n]$ given that $\{x_n\}_{n=1}^{\infty}$ is a basis of $[x_n]$. To prove this, we first show that the operator A maps $[x_n]$ into $[y_n]$ biuniquely.

Let $x \in [x_n]$. Then x can be expressed as $x = \sum_{i=1}^{\infty} a_i x_i$, a_i are scalars.

$$\therefore \quad A(x) = \sum_{i=1}^{\infty} a_i A(x_i) = \sum_{i=1}^{\infty} a_i y_i.$$

So, $A(x) \in [y_n]$.

Conversely, let $y \in [y_n]$. If y is not a limit point of the linear manifold generated by $\{y_n\}_{n=1}^{\infty}$, then y can be expressed as $y = \sum_{i=1}^{\infty} b_i y_i$, b_i are scalars. Since A^{-1} is continuous (Taylor 1957, p. 192), we get

$$A^{-1}(y) = \sum_{i=1}^{\infty} b_i A^{-1}(y_i) = \sum_{i=1}^{\infty} b_i x_i.$$

$$\therefore \quad A^{-1}(y) \in [x_n].$$

If, however, y is a limit point of the linear manifold spanned by $\{y_n\}_{n=1}^{\infty}$, then let $z_n \to y$ as $n \to \infty$, where each z_n can be expressed as $z_n = \sum_{i=1}^{\infty} c_i^{(n)} y_i$.

Consequently, $A^{-1}(z_n) = p_n$, say, belongs to $[x_n]$. Since $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence and A^{-1} is continuous, $\{p_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $[x_n]$. Now, $[x_n]$ is a closed linear manifold in the Banach space X. Consequently, $\{p_n\}_{n=1}^{\infty}$ converges to p, say, which belongs to $[x_n]$. Continuity of A^{-1} then gives

$$A^{-1}(y) = A^{-1}\left(\operatorname*{lt}_{n\to\infty} z_n\right) = \operatorname*{lt}_{n\to\infty} (A^{-1} z_n) = \operatorname*{lt}_{n\to\infty} p_n = p.$$

Consequently $A^{-1}(y) \in [x_n]$. Thus, the operator A maps $[x_n]$ into $[y_n]$ biuniquely. Considering now $[x_n]$ and $[y_n]$ as Banach spaces, it follows from the method of the proof of Lemma 2 that $\{y_n\}_{n=1}^{\infty}$ is a basis in $[y_n]$. So $\{y_n\}_{n=1}^{\infty}$ is a basic sequence in Y. This proves the lemma.

3. THEOREMS

Theorem 1—Let X be a reflexive Banach space with a basis and Y a normed linear space. Let A be a continuous linear operator on X onto Y possessing inverse. Then Y is also a reflexive Banach space with a basis.

PROOF: It follows from Lemmas 1 and 2 that Y is a Banach space with a basis. Let $\{\zeta_n\}_{n=1}^{\infty}$ be a basic sequence in Y. Suppose that $\left\{\sum_{i=1}^{n} a_i \zeta_i\right\}_{n=1}^{\infty}$ is bounded. Then there exists a constant K such that

$$\| a_1\zeta_1 + \ldots + a_n\zeta_n \| \leqslant k, \quad n = 1, 2, \ldots .$$

Let

$$A^{-1}(\zeta_n) = x_n, \quad n = 1, 2, \ldots .$$

Then by Lemma 3, $\{x_n\}$ is a basic sequence in X and so, by Theorem A, is boundedly complete.

Now,

$$\begin{aligned} \| a_1x_1+\ldots+a_nx_n \| &= \| a_1A^{-1}(\zeta_1)+\ldots+a_nA^{-1}(\zeta_n) \| \\ &= \| A^{-1}(a_1\zeta_1+\ldots+a_n\zeta_n) \| \\ &\leqslant \| A^{-1} \| \| a_1\zeta_1+\ldots+a_n\zeta_n \| \\ &\leqslant \| A^{-1} \| k, \text{ for } n = 1, 2, \ldots . \end{aligned}$$

This shows that the sequence $\left\{ \sum_{i=1}^{n} a_ix_i \right\}_{n=1}^{\infty}$ is bounded in X and consequently $\sum_{i=1}^{\infty} a_ix_i = x \in X$.

This implies that $A(x) = \sum_{i=1}^{\infty} a_i\zeta_i$ and so $\{\zeta_n\}_{n=1}^{\infty}$ is boundedly complete. Since $\{\zeta_n\}_{n=1}^{\infty}$ is arbitrary basic sequence in Y, by Theorem A it follows that Y is reflexive. This proves the theorem.

Let now X and Y be two reflexive Banach spaces and X'' and Y'' be the corresponding second normed conjugate spaces. Let J and J' be the cannonical mappings (Taylor 1957, p. 192) on X and Y onto X'' and Y'' respectively. It is known that J and J' are continuous linear operators.

Let $x'' \in X''$ and $x = J^{-1}(x'') \in X$. Let A be a continuous linear operator on X onto Y possessing inverse. Let $A(x) = y$ and $J'(y) = y''$. Then for any scalar α, $A(\alpha x) = \alpha A(x) = \alpha y$ and $J'(\alpha y) = \alpha y''$.

We now define an operator T mapping X'' into Y'' by the relation

$$T(x'') = \alpha y''.$$

Theorem 2—The operator T is a continuous linear operator mapping X'' onto Y''. If $\alpha \neq 0$ then T^{-1} exists.

PROOF: Let x''_1, $x''_2 \in X''$ and let $x_1 = J^{-1}(x''_1)$, $x_2 = J^{-1}(x''_2)$, $A(x_1) = y_1$, $A(x_2) = y_2$ and $J'(y_1) = y''_1$, $J'(y_2) = y''_2$.

Then $x_1+x_2 = J^{-1}(x''_1+x''_2)$, $A(x_1+x_2) = y_1+y_2$ and $J'(y_1+y_2) = y''_1+y''_2$.

So $$T(x''_1+x''_2) = \alpha(y''_1+y''_2) = \alpha y''_1+\alpha y''_2 = T(x''_1)+T(x''_2)$$

and for scalar β we have

$$T(\beta x''_1) = \alpha J'A(\beta x_1) = \beta\alpha J'A(x_1) = \beta T(x''_1).$$

So, T is linear. Also

$$\begin{aligned} \| T(x'') \| &= \| \alpha y'' \| = |\alpha| \| y'' \| = |\alpha| \| y \|, \text{ since } \| J'(y) \| = \| y \| \\ &= |\alpha| \| A(x) \| \leqslant |\alpha| \| A \| \| x \| \\ &= |\alpha| \| A \| \| x'' \|, \text{ since } \| J(x) \| = \| x \| \end{aligned}$$

where we suppose $J(x) = x''$, $A(x) = y$, $J'(y) = y''$. So the linear operator T is continuous and $\| T \| \leqslant | \alpha | \, \| A \|$.

Let further that $T(x'') = \alpha y'' = \theta$ and suppose, if possible, $x'' \neq \theta$. Then since A^{-1} and J'^{-1} exists we obtain successively that

$$x \neq \theta, \; y = A(x) \neq \theta, \; \alpha y \neq \theta, \text{ if } \alpha \neq 0, \; \alpha y'' \neq \theta.$$

This contradiction shows that $x'' = \theta$ and so T^{-1} exists. This completes the proof.

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