# STEADY FLOW OF A SECOND-ORDER FLUID IN THE ENTRANCE REGION OF A STRAIGHT CHANNEL

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The problem of the steady flow of a second-order fluid in the entrance region of a straight channel is investigated by approximating the boundary layer velocity profile by a cubic curve or a fourth degree curve. The analytical expression for pressure is obtained both by the principle of kinetic energy end-correction and by macroscopic mechanical energy balance for comparison. The expression for the inlet length is also obtained and it is observed that the non-Newtonian parameter is to increase the inlet length.

## 1. Introduction

The fluid enters a channel with a nearly uniform velocity distribution. The thickness of the boundary layer is zero at the entrance increasing along the channel. Retardation of the fluid in the boundary layer is accompanied by an acceleration of the central core in accordance with the requirement of continuity and there is a corresponding fall in pressure. After travelling a certain distance known as the entrance length the flow is said to be fully developed at all points and the boundary layer converges on the centre-line. An accurate description of the velocities and pressure within the entrance region is needed to calculate pressure drop for inlets and to correct data from viscometer studies. This correction is discussed by Goldstein (1938) for Newtonian fluids by assuming that the dissipation of energy in the inlet length is same as the dissipation in the same length when the velocity distribution is fully developed. This approximation is known as 'kinetic energy end-correction'. This problem is discussed by Bhatnagar and Lakshmana Rao (1957) for a class of Reiner-Rivlin fluids, Kapur and Gupta (1963, 1964*a*, *b*) for Power law fluids and Reiner-Philippoff fluids.

In the present paper we have discussed the inlet length problem for the flow of second-order fluid in a channel bounded by two semi-infinite parallel plates. The analytical expressions are obtained for (i) the inlet length, (ii) velocity profiles in the inlet length region and (iii) the pressure drop. Campbell and Slattery (1963) have determined the pressure drop for Newtonian fluids by taking into account the viscous dissipation within the boundary layer by using the technique of 'macroscopic mechanical energy balance'.

We have determined the pressure drop by using both the methods, i.e. kinetic energy end-correction and macroscopic mechanical energy blalance, for comparison. It is believed that the latter method gives a better flow description.

In the first section we have derived the boundary layer equations for second-order fluid by assuming asymptotic series in Reynolds number for the velocity components and pressure and by collecting the equations of zeroth order. The boundary layer equations thus obtained are integrated for the entrance region by Karman-Pohlhausen's method. The cubic and fourth degree velocity profiles are assumed. Following Kapur and Gupta (1963), the constants are determined by using the boundary conditions except one free constant which is obtained from the principle of equal integrated kinetic energies or the principle of least squares.

## 2. Equations of Motion

The constitutive equation of an incompressible second-order fluid has been given by Coleman and Noll (1960) as

$$\tau_{ij} = \phi_1 A_{(1)ij} + \phi_2 A_{(2)ij} + \phi_3 A_{(1)ik} A_{(2)kj} \quad \ldots \quad (2.1)$$

where

$$A_{(1)ij} = v_{i,j} + v_{j,i} \quad \ldots \quad (2.2)$$

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2v_{m,i}\, v_{m,j} \quad \ldots \quad (2.3)$$

and

$$S_{ij} = \tau_{ij} - pg_{ij} \quad \ldots \quad (2.4)$$

such that $S_{ij}$ is the stress tensor; $g_{ij}$ the metric tensor, $v_i$ and $a_i$ the velocity and acceleration vectors respectively; $p$ the pressure; $\phi_1$, $\phi_2$ and $\phi_3$ material constants—the coefficient of ordinary viscosity, the coefficient of visco-elasticity and the coefficient of cross-viscosity respectively.

The momentum equations for the steady incompressible flow are

$$\rho v^j v_{i,j} = -p_{,i} + \tau^j_{i,j} \quad \ldots \quad (2.5)$$

and the equation of continuity is

$$v^i_{,i} = 0 \quad \ldots \quad (2.6)$$

where $v_i$ is the velocity vector; $\tau^j_i$ is the stress tensor; $p$ the pressure; $\rho$ is the density; and comma denotes covariant differentiation. Transforming all tensor components in eqns. (2.1) to (2.6) into physical components in Cartesian coordinates, we get the following equations for steady two-dimensional flow. Equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \ldots \quad (2.7)$$

Momentum equations:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\phi_1\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$
$$+\phi_2\left[\frac{2\partial u}{\partial x}\left(\frac{5\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{3\partial u}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right.$$
$$+\frac{2\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{2\partial v}{\partial x}\right)+\frac{\partial v}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{3\partial v}{\partial x}\right)$$
$$\left.+u\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right]$$
$$+\phi_3\left[8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right] \quad \ldots (2.8)$$

and

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}+\phi_1\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$
$$+\phi_2\left[2\frac{\partial v}{\partial y}\left(5\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right)+3\frac{\partial v}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right.$$
$$+2\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+2\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(3\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$
$$\left.+u\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right)\right]$$
$$+\phi_3\left[8\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]. \quad \ldots (2.9)$$

Let us introduce the following non-dimensional quantities

$$t' = \frac{t}{T},\ x' = \frac{x}{X},\ y' = \frac{y}{Y},\ u' = \frac{u}{U},\ v' = \frac{v}{V} \text{ and } p' = \frac{p}{P} \quad \ldots (2.10)$$

where $T$, $X$, $Y$, $U$, $V$ and $P$ are the units of measurement of the corresponding quantities.

Substituting (2.10) in (2.7) to (2.9), we get

$$\rho\left(u'\frac{\partial u'}{\partial x'}+\frac{XV}{YU}v'\frac{\partial u'}{\partial y'}\right) = -\frac{P}{U^2}\frac{\partial p'}{\partial x'}+\phi_1\left[\frac{1}{XU}\frac{\partial^2 u'}{\partial x'^2}+\frac{X}{Y^2 U}\frac{\partial^2 u'}{\partial y'^2}\right]$$
$$+\phi_2\left[\frac{2}{U}\frac{\partial u'}{\partial x'}\left(\frac{5U}{X^2}\frac{\partial^2 u'}{\partial x'^2}+\frac{U}{Y^2}\frac{\partial^2 u'}{\partial y'^2}\right)+\frac{3}{YU}\frac{\partial u'}{\partial y'}\frac{\partial}{\partial x'}\left(\frac{V}{X}\frac{\partial v'}{\partial x'}\right.\right.$$
$$\left.+\frac{U}{Y}\frac{\partial u'}{\partial y'}\right)+2\frac{V}{U^2 X}\frac{\partial v'}{\partial x'}\frac{\partial}{\partial x'}\left(\frac{U}{Y}\frac{\partial u'}{\partial y'}+2\frac{V}{X}\frac{\partial v'}{\partial x'}\right)$$
$$+\frac{XV}{U^2 Y^2}\frac{\partial v'}{\partial y'}\frac{\partial}{\partial y'}\left(\frac{U}{Y}\frac{\partial u'}{\partial y'}+\frac{3V}{X}\frac{\partial v'}{\partial x'}\right)+\frac{u'}{U}\frac{\partial}{\partial x'}\left(\frac{U}{X^2}\frac{\partial^2 u'}{\partial x'^2}\right.$$
$$\left.\left.+\frac{U}{Y^2}\frac{\partial^2 u'}{\partial y'^2}\right)+\frac{XV}{U^2 Y}v'\frac{\partial}{\partial y'}\left(\frac{U}{X^2}\frac{\partial^2 u'}{\partial x'^2}+\frac{U}{Y^2}\frac{\partial^2 u'}{\partial y'^2}\right)\right]$$
$$+\phi_3\left[\frac{8}{X^2}\frac{\partial u'}{\partial x'}\frac{\partial^2 u'}{\partial x'^2}+2\left(\frac{1}{UY}\frac{\partial u'}{\partial y'}+\frac{V}{XU^2}\frac{\partial v'}{\partial x'}\right)\frac{\partial}{\partial x'}\left(\frac{U}{Y}\frac{\partial u'}{\partial y'}+\frac{V}{X}\frac{\partial v'}{\partial x'}\right)\right], \quad \ldots (2.11)$$

$$\rho\left(u'\frac{\partial v'}{\partial x'}+\frac{XV}{YU}v'\frac{\partial v'}{\partial y'}\right) = -\frac{PX}{UVY}\frac{\partial p'}{\partial y'}+\phi_1\left[\frac{1}{UX}\frac{\partial^2 v'}{\partial x'^2}+\frac{X}{UY^2}\frac{\partial^2 v'}{\partial y'^2}\right]$$

$$+\phi_2\left[2\frac{\partial v'}{\partial y'}\left(\frac{5VX}{UY^3}\frac{\partial^2 v'}{\partial y'^2}+\frac{V}{XYU}\frac{\partial^2 v'}{\partial x'^2}\right)\right.$$

$$+3\frac{\partial v'}{\partial x'}\frac{\partial}{\partial y'}\left(\frac{V}{UXY}\frac{\partial v'}{\partial x'}+\frac{1}{Y^2}\frac{\partial u'}{\partial y'}\right)+2\frac{\partial u'}{\partial y'}\frac{\partial}{\partial y'}\left(\frac{1}{Y^2}\frac{\partial v'}{\partial x'}\right.$$

$$\left.+\frac{2XU}{VY^3}\frac{\partial u'}{\partial y'}\right)+\frac{\partial u'}{\partial x'}\frac{\partial}{\partial x'}\left(3\frac{U}{XYV}\frac{\partial u'}{\partial y'}+\frac{1}{X^2}\frac{\partial v'}{\partial x'}\right)$$

$$+u'\frac{\partial}{\partial x'}\left(\frac{1}{Y^2}\frac{\partial^2 v'}{\partial y'^2}+\frac{1}{X^2}\frac{\partial^2 v'}{\partial x'^2}\right)+v'\frac{\partial}{\partial y'}\left(\frac{XV}{UY^3}\frac{\partial^2 v'}{\partial y'^2}\right.$$

$$\left.\left.+\frac{V}{XUY}\frac{\partial^2 v'}{\partial x'^2}\right)\right]+\phi_3\left[\frac{8VX}{Y^3U}\frac{\partial v'}{\partial y'}\frac{\partial^2 v'}{\partial y'^2}+2\left(\frac{X}{VY^2}\frac{\partial u'}{\partial y'}\right.\right.$$

$$\left.\left.+\frac{1}{UY}\frac{\partial v'}{\partial x'}\right)\frac{\partial}{\partial y'}\left(\frac{U}{Y}\frac{\partial u'}{\partial y'}+\frac{V}{X}\frac{\partial v'}{\partial x'}\right)\right] \qquad \text{(2.12)}$$

and

$$\frac{\partial u'}{\partial x'}+\frac{XV}{YU}\frac{\partial v'}{\partial y'}=0. \qquad \text{(2.13)}$$

We will regard $X$ and $U$ as the fundamental units of measurements and construct with them the Reynolds number of the flow, viz.

$$R=\frac{XU\rho}{\phi_1}. \qquad \text{(2.14)}$$

Now the unit of measurement of time and pressure in terms of $X$ and $U$ can be represented as

$$T=\frac{X}{U} \text{ and } P=\rho U^2. \qquad \text{(2.15)}$$

We have yet to determine the units of measurement of $Y$ and $V$. They are determined by the requirement that the system of eqns. (2.11) to (2.13) must have only a single flow parameter, since the flow past geometrically similar bodies is geometrically similar, namely the Reynolds number defined by (2.14). For this reason, as can be seen easily, we must substitute

$$\frac{XV}{YU}=1 \text{ and } \frac{\nu_1 X}{Y^2U}=1; \quad \nu_1=\frac{\phi_1}{\rho} \qquad \text{(2.16)}$$

or

$$Y=\frac{X}{\sqrt{R}} \text{ and } V=\frac{U}{\sqrt{R}}. \qquad \text{(2.17)}$$

The system of eqns. (2.11) to (2.13) reduce to

$$\left(u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right) = -\frac{\partial p'}{\partial x'}+\frac{1}{R}\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}+K\left[2\frac{\partial u'}{\partial x'}\left(5\frac{\partial^2 u}{\partial x'^2}+R\frac{\partial^2 u'}{\partial y'^2}\right)\right.$$
$$+3\frac{\partial u'}{\partial y'}\frac{\partial}{\partial x'}\left(\frac{\partial v'}{\partial x'}+R\frac{\partial u'}{\partial y'}\right)+2\frac{\partial v'}{\partial x'}\frac{\partial}{\partial x'}\left(\frac{\partial u'}{\partial y'}+\frac{2}{R}\frac{\partial v'}{\partial x'}\right)$$
$$+\frac{\partial v'}{\partial y'}\frac{\partial}{\partial y'}\left(R\frac{\partial u'}{\partial y'}+3\frac{\partial v'}{\partial x'}\right)+u'\frac{\partial}{\partial x'}\left(\frac{\partial^2 u'}{\partial x'^2}+R\frac{\partial^2 u'}{\partial y'^2}\right)$$
$$\left.+v'\frac{\partial}{\partial y'}\left(\frac{\partial^2 u'}{\partial x'^2}+R\frac{\partial^2 u'}{\partial y'^2}\right)\right]+S\left[8\frac{\partial u'}{\partial x'}\frac{\partial^2 u'}{\partial x'^2}\right.$$
$$\left.+2\left(\sqrt{R}\frac{\partial u'}{\partial y'}+\frac{1}{\sqrt{R}}\frac{\partial v'}{\partial x'}\right)\frac{\partial}{\partial x'}\left(\sqrt{R}\frac{\partial u'}{\partial y'}+\frac{1}{\sqrt{R}}\frac{\partial v'}{\partial x'}\right)\right], \quad (2.18)$$

$$\frac{1}{R}\left(u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right) = -\frac{\partial p}{\partial y'}+\frac{1}{R^2}\frac{\partial^2 v'}{\partial x'^2}+\frac{1}{R}\frac{\partial^2 v'}{\partial y'^2}+K\left[2\frac{\partial v'}{\partial y'}\left(5\frac{\partial^2 v'}{\partial y'^2}+\frac{1}{R}\frac{\partial^2 v'}{\partial x'^2}\right)\right.$$
$$+3\frac{\partial v'}{\partial x'}\frac{\partial}{\partial y'}\left(\frac{1}{R}\frac{\partial v'}{\partial x'}+\frac{\partial u'}{\partial y'}\right)$$
$$+2\frac{\partial u'}{\partial y'}\frac{\partial}{\partial y'}\left(\frac{\partial v'}{\partial x'}+2R\frac{\partial u'}{\partial y'}\right)+\frac{\partial u'}{\partial x'}\frac{\partial}{\partial x'}\left(3\frac{\partial u'}{\partial y'}+\frac{1}{R}\frac{\partial v'}{\partial x'}\right)$$
$$\left.+u'\frac{\partial}{\partial x'}\left(\frac{1}{R}\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right)+v'\frac{\partial}{\partial y'}\left(\frac{\partial^2 v'}{\partial y'^2}+\frac{1}{R}\frac{\partial^2 v'}{\partial x'^2}\right)\right]$$
$$+S\left[8\frac{\partial v'}{\partial y'}\frac{\partial^2 v'}{\partial y'^2}+2\left(\frac{\partial u'}{\partial y'}+\frac{1}{R}\frac{\partial v'}{\partial x'}\right)\frac{\partial}{\partial y'}\left(R\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right)\right] \quad (2.19)$$

and

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'} = 0 \qquad \ldots \quad (2.20)$$

where $K$ and $S$ are the dimensionless parameters characterizing respectively the visco-elasticity and cross-viscosity of the fluid.

We develop the solution of the above equations formally with the help of a power series for large Reynolds number.

Let

$$\left.\begin{aligned} u' &= u_0+\frac{u_1}{\sqrt{R}}+\frac{u_2}{(\sqrt{R})^2}+\ldots \\ v' &= v_0+\frac{v_1}{\sqrt{R}}+\frac{v_2}{(\sqrt{R})^2}+\ldots \\ p' &= p_0+\frac{p_1}{\sqrt{R}}+\frac{p_2}{(\sqrt{R})^2}+\ldots \end{aligned}\right\} \qquad \ldots \quad (2.21)$$

where $u_0$, $v_0$, $p_0$, $u_1$, $v_1$, $p_1 \ldots$ are the functions of the dimensionless co-ordinates. We further assume

$$\left.\begin{aligned} K &= \frac{K_0}{R} \\ S &= \frac{S_0}{R} \end{aligned}\right\} \qquad \ldots \qquad (2.22)$$

Substituting (2.21) and (2.22) in (2.18) to (2.20) and collecting the equations of zeroth order, we get

$$u_0\frac{\partial u_0}{\partial x'}+v_0\frac{\partial u_0}{\partial y'} = -\frac{\partial}{\partial x'}\left[p_0-(2K_0+S_0)\left(\frac{\partial u_0}{\partial y'}\right)^2\right]+\frac{\partial^2 u_0}{\partial y'^2}$$

$$+K_0\left[\frac{\partial u_0}{\partial x'}\frac{\partial^2 u_0}{\partial y'^2}+u_0\frac{\partial^3 u_0}{\partial x'\partial y'^2}+v_0\frac{\partial^3 u_0}{\partial y'^3}+\frac{\partial u_0}{\partial y'}\frac{\partial^2 v_0}{\partial y'^2}\right], \qquad (2.23)$$

$$\frac{\partial}{\partial y'}\left[p_0-(2K_0+S_0)\left(\frac{\partial u_0}{\partial y'}\right)^2\right] = 0, \qquad \ldots \qquad (2.24)$$

$$\frac{\partial u_0}{\partial x'}+\frac{\partial v_0}{\partial y'} = 0 \qquad \ldots \qquad (2.25)$$

Reverting eqns. (2.23) to (2.25) to the dimensional form and dropping the index null, we get the boundary layer equations for two-dimensional viscous incompressible steady flow of second order fluid as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu_1\frac{\partial^2 u}{\partial y^2}+\nu_2\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+u\frac{\partial^3 u}{\partial x\partial y^2}\right.$$

$$\left.+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right) = 0, \qquad \ldots \qquad (2.26)$$

$$\frac{\partial P}{\partial y} = 0 \qquad \ldots \qquad (2.27)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \qquad \ldots \qquad (2.28)$$

where pressure $P$ is a modified pressure (Bhatnagar 1961) given by

$$\frac{P}{\rho} = \frac{p}{\rho}-(2\nu_2+\nu_3)\left(\frac{\partial u}{\partial y}\right)^2; \quad \nu_2 = \frac{\phi_2}{\rho},\ \nu_3 = \frac{\phi_3}{\rho}, \qquad \ldots \qquad (2.29)$$

and if $p_0$ be the pressure in the free stream, then

$$\frac{P}{\rho} = \frac{p}{\rho}-(2\nu_2+\nu_3)\left(\frac{\partial u}{\partial y}\right)^2 = \frac{p_0}{\rho}, \qquad \ldots \qquad (2.30)$$

so that the pressure in the boundary layer is given by

$$\frac{p}{\rho} = \frac{p_0}{\rho}+(2\nu_2+\nu_3)\left(\frac{\partial u}{\partial y}\right)^2. \qquad \ldots \qquad (2.31)$$

And if $U$ be the velocity in the free stream, then

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = U\frac{\partial U}{\partial x}. \qquad \ldots \qquad (2.32)$$

With the help of relation (2.32), (2.26) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x}+\nu_1\frac{\partial^2 u}{\partial y^2}+\nu_2\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+u\frac{\partial^3 u}{\partial x\partial y^2}\right.$$
$$\left.+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right) = 0. \qquad \ldots \qquad (2.33)$$

## 3. Boundary Layer Velocity Profiles

Boundary layer velocity profiles may be approximated by a third (Bogue 1959) or fourth degree curve. The cubic velocity profile contains four unknown constants of which three have been determined from certain boundary conditions and the remaining one from the principle of least square or from the principle of equal integrated kinetic energies.

Kapur and Gupta (1963) have discussed two fourth degree velocity profiles, one of which reduces to parabolic profile for Newtonian fluids in the fully developed state. Thus we have two boundary layer velocity profiles

$$\frac{u}{U} = a\left(\frac{h-y}{\delta}\right)+(3-2a)\left(\frac{h-y}{\delta}\right)^2+(a-2)\left(\frac{h-y}{\delta}\right)^3 \qquad \ldots \qquad (3.1)$$

and

$$\frac{u}{U} = a\left(\frac{h-y}{\delta}\right)+\left(\frac{7-5a}{3}\right)\left(\frac{h-y}{\delta}\right)^2$$
$$+\left(\frac{a-2}{3}\right)\left(\frac{h-y}{\delta}\right)^3+\left(\frac{a-2}{3}\right)\left(\frac{h-y}{\delta}\right)^4. \qquad \ldots \qquad (3.2)$$

Profiles given by eqns. (3.1) and (3.2) reduce to parabolic profiles in the fully developed state, which are used in our analysis.

Another fourth degree profile, satisfying certain boundary conditions, is given as

$$\frac{u}{U} = a\left(\frac{h-y}{\delta}\right)+(6-3a)\left(\frac{h-y}{\delta}\right)^2$$
$$+(3a-8)\left(\frac{h-y}{\delta}\right)^3+(3-a)\left(\frac{h-y}{\delta}\right)^4 \qquad \ldots \qquad (3.3)$$

in which additional boundary condition has been taken as vanishing of the second derivative of the velocity at the edge of the boundary layer, in order to determine additional constant, but this velocity profile does not reduce to parabolic form in the fully developed state.

## 4. Boundary Layer Equations and Their Integration

### (a) *Third Degree Velocity Profile*

Integrating (2.33) with respect to $y$ between the limits 0 to $\delta$ ($\delta$ being the boundary layer thickness) and using Schlichting's (1960) simplification, we have

$$\int_0^\delta \left(2u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{\partial U}{\partial x}\right)dy = \int_0^\delta \nu_1 \frac{\partial^2 u}{\partial y^2}dy + \nu_2\int_0^\delta \left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right)dy. \tag{4.1}$$

On using the substitution $y = h - y'$ in (4.1), we have

$$\int_{h-\delta}^{h} \left(2u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{\partial U}{\partial x}\right)dy = \nu_1\left(\frac{\partial u}{\partial y}\right)_{y=h} + \nu_2\left[\int_{h-\delta}^{h}\left(3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} + u\frac{\partial^3 u}{\partial x\partial y^2}\right)dy + \left(v\frac{\partial^2 u}{\partial y^2}\right)_{y=h-\delta}\right]. \tag{4.2}$$

From (3.1) and (4.2), we get

$$\left(B_1 + \frac{\nu_2 D_1}{\delta^2}\right)\delta\frac{dU}{dx} + \left(B_2 + \frac{\nu_2 D_2}{\delta^2}\right)U\frac{d\delta}{dx} = \frac{\nu_1 a}{\delta} \quad \dots \quad \dots \tag{4.3}$$

where

$$B_1 = \frac{318 - 17a - 8a^2}{420}, \quad \dots \quad \dots \quad \dots \quad \dots \tag{4.4}$$

$$B_2 = \frac{9a + 54 - 4a^2}{420}, \quad \dots \quad \dots \quad \dots \quad \dots \tag{4.5}$$

$$D_1 = \frac{3a - 7a^2 - 18}{10} \quad \dots \quad \dots \quad \dots \quad \dots \tag{4.6}$$

and

$$D_2 = \frac{a^2 - 9a + 54}{10}. \quad \dots \quad \dots \quad \dots \quad \dots \tag{4.7}$$

From the continuity equation, we have

$$V\,.\,2h = U(2h - 2\delta) + 2\int_{h-\delta}^{h} u\,dy. \quad \dots \quad \dots \quad \dots \tag{4.8}$$

Now, from (3.1) and (4.8), we get

$$U = \frac{V}{1 - kZ} \quad \dots \quad \dots \quad \dots \quad \dots \tag{4.9}$$

where $Z = \frac{\delta}{h}$, $k = \frac{1}{2} - \frac{a}{12}$ and $V$ is the entry velocity.

Substituting for $U$ in (4.3), we obtain

$$\frac{x/h}{R/a} = \int_0^{\delta/h}\left[\left(\frac{B_1 k Z^2}{(1-kZ)^2} + \frac{B_2 Z}{(1-kZ)}\right) + \frac{\nu_2}{h^2}\left(\frac{D_1 k}{(1-kZ)^2} + \frac{D_2}{Z(1-kZ)}\right)\right]dZ \tag{4.10}$$

where Reynolds number,

$$R = \frac{Vh\rho}{\phi_1}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (4.11)$$

The inlet length $L$ can be obtained when $\delta = h$.

From equation (4.10), it is seen that for a given $\delta$, $x$ and $L$ have increased as compared with the Newtonian fluid of viscosity coefficient $\phi_1$.

(b) *Fourth Degree Velocity Profile*

In order to integrate (4.2), we make use of the fourth degree velocity profile (3.2) and on simplification, we have

$$\left(C_1+\frac{\nu_2 E_1}{\delta^2}\right)\delta\frac{dU}{dx}+\left(C_2+\frac{\nu_2 E_2}{\delta^2}\right)U\frac{d\delta}{dx} = \frac{\nu_1 a}{\delta} \qquad \ldots \quad \ldots \ (4.12)$$

where

$$C_1 = \frac{8822-461a-274a^2}{11340}, \qquad \ldots \quad \ldots \quad \ldots \ (4.13)$$

$$C_2 = \frac{1450+305a-137a^2}{11340}, \qquad \ldots \quad \ldots \quad \ldots \ (4.14)$$

$$E_1 = \frac{813a-1668a^2-2514}{1890}, \qquad \ldots \quad \ldots \quad \ldots \ (4.15)$$

and

$$E_2 = \frac{2238+24a^2-411a}{378}. \qquad \ldots \quad \ldots \quad \ldots \ (4.16)$$

From the equation of continuity, we get

$$U = \frac{V}{1-mZ} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (4.17)$$

where

$$Z = \frac{\delta}{h} \text{ and } m = \frac{47}{90}-\frac{17}{180}a.$$

Substituting for $U$ in (4.12), we get

$$\frac{x/h}{R/a} = \int_0^{\delta/h}\left[\left(C_1\frac{mZ^2}{(1-mZ)^2}+C_2\frac{Z}{(1-mZ)}\right)+\frac{\nu_2}{h^2}\left(\frac{E_1 m}{(1-mZ)^2}+\frac{E_2}{Z(1-mZ)}\right)\right]dZ. \qquad \ldots \ (4.18)$$

The inlet length $L$ can be obtained as in the preceding section when $\delta = h$. It may be observed from (4.18) that for a given $\delta$, $x$ and $L$ have increased as compared with the Newtonian fluid of viscosity coefficient $\phi_1$.

## 5. Kinetic Energy End-correction

Let $p_0$ and $p_1$ be the pressures at the entry and at the end of the inlet length respectively. Let $V$ be the entry velocity, $2h$ the channel width, $u$

the velocity of any fluid particle along the channel, then from the equation of continuity, we get

$$V \,.\, 2h = \int_0^h u \,.\, 2dy. \qquad (5.1)$$

The rate at which pressures are doing work per unit width of the wall is

$$2hp_0 V - 2\int_0^h p_1 u \, dy. \qquad (5.2)$$

The rate of inflow of kinetic energy per unit width is $h\rho V^3$ and the rate of outflow at the end of the inlet length is

$$2\int_0^h \tfrac{1}{2}\rho u^3 \, dy = h\rho U^3 \text{ (say)} \qquad (5.3)$$

so that the excess of the rate of outflow of kinetic energy over the rate of inflow is

$$h\rho(U^3 - V^3). \qquad (5.4)$$

The rate of dissipation of energy is

$$\int\int \tau_{xy} e_{xy} \, dx \, dy \qquad (5.5)$$

where the integration is taken over the inlet length. For the end-correction, we assume that the dissipation of energy in the inlet length is the same as the dissipation in the same length after the end of the inlet length, i.e. when velocity profile is fully developed and parabolic. Thus the rate at which work is done in changing the volume and the shape of the element from (5.5) is

$$= 2L\int_0^h \tau_{xy} e_{xy} \, dy, \qquad (5.6)$$

$e_{xy}$ is obtained from $\dfrac{\partial u}{\partial y}$ for the fully developed velocity profile and $\tau_{xy}$ is then obtained from the empirical relation for the fluid. Finally, equating the rate of work done by the pressures to the total rate of dissipation of energy, we get the expression for the pressure drop obtained from kinetic energy end-correction

$$p_0 - \left( \frac{\int_0^h p_1 u \, dy}{\int_0^h u \, dy} \right) = \tfrac{1}{2}\rho V^2 \left( -1 + \frac{U^3}{V^3} \right) + \frac{L}{Vh}\int_0^h \tau_{xy} e_{xy} \, dy. \qquad (5.7)$$

In case of second-order fluid, fully developed velocity profile is same as that of Newtonian fluid of viscosity coefficient $\phi_1$, i.e.

$$u = \frac{\lambda}{2\phi_1}(h^2 - y^2), \qquad (5.8)$$

where $\lambda$ is the constant pressure gradient and pressure is given by

$$p = -\lambda x + p_{00} + (2\phi_2+\phi_3)\left(\frac{\lambda y}{\phi_1}\right)^2. \qquad (5.9)$$

Let the origin of coordinates be at the end of inlet length, then

$$p_1 = p_{00} + (2\phi_2+\phi_3)\left(\frac{\lambda y}{\phi_1}\right)^2. \qquad (5.10)$$

From (5.7) to (5.10), we have

$$\frac{p_0-p_{00}}{\frac{1}{2}\rho V^2} = \frac{19}{35} + \frac{L\lambda}{\frac{1}{2}\rho V^2} + \frac{2\lambda^2 h^2}{5\rho V^2 \phi_1{}^2}(2\phi_2+\phi_3). \qquad (5.11)$$

## 6. Method of Macroscopic Mechanical Energy Balance

The macroscopic mechanical energy balance (Bird *et al.* 1960) is written as

$$\Delta\left(\tfrac{1}{2}\rho S < u^3 >\right) + w\int_{p_1}^{p_2}\frac{dp}{\rho} + E_v = 0. \qquad (6.1)$$

Here $E_v$ is the total rate of viscous dissipation of energy and $w$ is the mass rate of flow. While writing (6.1) it is understood that the square of velocity component parallel to the channel is larger than the sum of the squares of all other velocity components.

Applying (6.1) to the fluid between the inlet and any arbitrary cross-section, we get

$$\tfrac{1}{2}\rho\int_0^h u^3\,dy - \tfrac{1}{2}\rho V^3 h + p\int_0^h u\,dy - p_0 V h + E_v = 0. \qquad (6.2)$$

$E_v$, the total rate of dissipation of energy, is given by

$$E_v = \int_0^X\int_0^h \tau_{ij} e_{ij}\,dy\,dx. \qquad (6.3)$$

When all velocity derivatives are neglected with respect to $\frac{\partial u}{\partial y}$,

$$E_v = \int_0^X\int_0^h\left[\phi_1\left(\frac{\partial u}{\partial y}\right) + \phi_2\left(u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial x} + v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right)\right]dy\,dx. \qquad (6.4)$$

Eliminating $E_v$ between (6.2) and (6.4) and differentiating the resulting equation with respect to the axial distance, we obtain

$$Vh\frac{dp}{dx} = -\tfrac{1}{2}\rho\frac{d}{dx}\int_0^h u^3\,dy - \int_0^h\left[\phi_1\left(\frac{\partial u}{\partial y}\right)^2 + \phi_2\left(u\frac{\partial^2 u}{\partial y\partial x}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2\right)\right]dy. \qquad (6.5)$$

Now in order to evaluate (6.5), we use the velocity distribution

$$\left.\begin{aligned} \frac{u}{U} &= 1, \quad \text{for } y \le h-\delta, \\ \frac{u}{U} &= 2\left(\frac{h-y}{\delta}\right) - \left(\frac{h-y}{\delta}\right)^2, \quad \text{for } y \ge h-\delta. \end{aligned}\right\} \qquad (6.6)$$

From (6.5) and (6.6), we have

$$-Vh\frac{dp}{dx}=\frac{27}{35}V^3h\rho\frac{(24-19Z)}{(3-Z)^4}\frac{dZ}{dx}+\frac{12\phi_1V^2}{h}\frac{1}{Z(3-Z)}$$
$$+\frac{36\phi_2V^3}{5h}\frac{(4Z-3)}{Z^2(3-Z)^4}\frac{dZ}{dx} \qquad \text{(6.7)}$$

where

$$Z=\frac{\delta}{h}.$$

On integration of (6.7), the pressure drop is given by

$$\frac{p_0-p}{\frac{1}{2}\rho V^2}=\int_0^Z\left[\frac{6}{35}(468Z^{-2}+39Z^{-1}-98)\right.$$
$$\left.+\frac{\nu_2}{h^2}\frac{72}{5}(87Z^{-2}-86Z^{-1}+70)\right](3-Z)^{-4}\,dZ. \qquad \text{(8.6)}$$

## 7. Determination of the Free Constant occurring in Velocity Profiles

The velocity profiles in the inlet length are

$$\frac{u}{V}=\frac{1}{(1-kZ)}\left[a\left(\frac{h-y}{\delta}\right)+(3-2a)\left(\frac{h-y}{\delta}\right)^2+(a-2)\left(\frac{h-y}{\delta}\right)^3\right] \qquad \text{(7.1)}$$

and

$$\frac{u}{V}=\frac{1}{(1-mZ)}\left[a\left(\frac{h-y}{\delta}\right)+\left(\frac{7-5a}{3}\right)\left(\frac{h-y}{\delta}\right)^2\right.$$
$$\left.+\left(\frac{a-2}{3}\right)\left(\frac{h-y}{\delta}\right)^3+\left(\frac{a-2}{3}\right)\left(\frac{h-y}{\delta}\right)^4\right]. \qquad \text{(7.2)}$$

Now in order to determine the value of the free constant $a$, we shall be applying the principle of least squares or principle of equal integrated kinetic energies.

### (a) *Principle of Least Squares*

Let $u$ be the fully developed velocity and let $u_1$ be the limiting velocity at the end of the inlet length, which is determined from (7.1) or (7.2) when $\delta = h$. We obtained the value of the free constant $a = 2$ on using the principle of least squares, i.e. on minimizing the integral

$$\int_0^h (u-u_1)^2\,dy. \qquad \text{(7.3)}$$

### (b) *Use of Principle of Equal Integrated Kinetic Energies*

According to the principle of equal integrated kinetic energies, we choose $a$ so that the integrated kinetic energy for the fully developed profile is the same as the integrated kinetic energy for the limiting velocity at the end of the inlet length, i.e. $a$ is to be chosen such that

$$\frac{1}{2}\int_0^h u^3\,dy = \frac{1}{2}\int_0^h u_1^3\,dy. \qquad (7.4)$$

The fully developed velocity profile is given by

$$u = \frac{3V}{2h^2}(h^2-y^2), \qquad (7.5)$$

and limiting velocities are given by

$$u_1 = \frac{12V}{(a+6)}\left[a\left(\frac{h-y}{h}\right)+(3-2a)\left(\frac{h-y}{h}\right)^2+(a-2)\left(\frac{h-y}{h}\right)^3\right] \qquad (7.6)$$

(for third degree profile)

and

$$u_1 = \frac{180V}{(86+17a)}\left[a\left(\frac{h-y}{h}\right)+\left(\frac{7-5a}{3}\right)\left(\frac{h-y}{h}\right)^2 +\left(\frac{a-2}{3}\right)\left(\frac{h-y}{h}\right)^3+\left(\frac{a-2}{3}\right)\left(\frac{h-y}{h}\right)^4\right] \qquad (7.7)$$

(for fourth degree profile).

Substituting the value of $u$ from (7.5) and $u_1$ from either of the eqns. (7.6) or (7.7) in (7.4), then on integration $a$ is given by

$$a^3-22a^2-152a+384 = 0. \qquad (7.8)$$

(for third degree velocity profile)

or

$$627323a^3-13024098a^2-72934584a+192946976 = 0 \qquad (7.9)$$

(for fourth degree velocity profile).


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