CLASSIFICATION OF ELECTROMAGNETIC FIELDS

by K. P. Singh *and* S. R. Roy, *Department of Mathematics, Banaras Hindu University, Varanasi 5*

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In this paper it has been shown that the electromagnetic field lends itself to a classification similar to the Petrov classification of the gravitational field. A certain tensor possessing all the algebraic properties of the conformal curvature tensor has been couched in terms of the electromagnetic field tensor. The Petrov classification of this tensor shows that the non-null electromagnetic field corresponds to Petrov type ID whereas the null electromagnetic field is always of type IIN.

1. Introduction

In the absence of an invariant formulation of gravitational energy the classification of the radiative and non-radiative types of gravitational fields has been done on the analogy of the electromagnetic field. This has been effected through the algebraic structure of the conformal curvature tensor C_{hijk} of the gravitational field. Herein we construct a tensor L_{hijk} for the electromagnetic field analogous to the conformal curvature tensor, the Petrov classification of which gives back the two types of electromagnetic fields, the null and non-null. It is found that according to this scheme the non-null electromagnetic fields correspond to Petrov type ID and the null electromagnetic fields are always of type IIN. The other Petrov types do not manifest themselves in this scheme.

2. The Tensor L_{hijk}

The electromagnetic field is given by an antisymmetric tensor F_{ij} which is connected with the metric of the Riemannian space through Einstein-Maxwell equations. We define a tensor P_{hijk} as follows:

$$P_{hijk} = 2F_{hi}F_{jk} - F_{hj}F_{ki} - F_{hk}F_{ij}. \qquad (1)$$

It can easily be seen that

$$\left.\begin{aligned} P_{[hi]jk} &= 0 \\ P_{hi[jk]} &= 0 \\ P_{hijk} &= P_{jkhi} \\ P_{h[ijk]} &= 0 \end{aligned}\right\}. \qquad (2)$$

The contracted tensor P_{ij} is given by

$$P_{ij} = g^{hk}P_{hijk} = -3F_{ir}F_j{}^r. \qquad (3)$$

If $P_{ij} = 0$ the electromagnetic field disappears. Hence we define another tensor

$$L_{hijk} = P_{hijk}+\tfrac{3}{2}[g_{hj}E_{ik}+g_{ik}E_{hj}-g_{hk}E_{ij}-g_{ij}E_{hk}] + \tfrac{1}{4}F_{rs}F^{rs}(g_{hk}g_{ij}-g_{hj}g_{ik}) \qquad (4)$$

where E_{ij} is the electromagnetic energy tensor given by

$$E_{ij} = -F_{ir}F_j{}^r+\tfrac{1}{4}F_{rs}F^{rs}g_{ij}. \qquad (5)$$

It is clear that L_{hijk} possesses all the algebraic properties of P_{hijk} indicated in (2) and also

$$L_{ij} = g^{hk}L_{hijk} = 0 \qquad (6)$$

identically.

It may be noted here that L_{hijk} is the only possible tensor quadratic in F_{ij} possessing the algebraic properties of the conformal curvature tensor C_{hijk}. Since the electromagnetic energy tensor is a quadratic expression in F_{ij} the tensor L_{hijk} is of the proper order in F_{ij} for effecting the classification.

3. The Electromagnetic Field

Usually the electromagnetic fields are divided into three classes:

$$\left.\begin{array}{ll} (a) & F_{ij}F^{ij} \neq 0, \quad \overset{*}{F}_{ij}F^{ij} \neq 0 \\ (b) & F_{ij}F^{ij} \neq 0, \quad \overset{*}{F}_{ij}F^{ij} = 0 \\ (c) & F_{ij}F^{ij} = 0, \quad \overset{*}{F}_{ij}F^{ij} = 0 \end{array}\right\} \qquad (7)$$

where

$$*F^{ij} = \tfrac{1}{2}\epsilon^{ijkl}F_{kl} \qquad (8)$$

ϵ^{ijkl} being the Levi-Civita tensor. Of these three types the first two are non-null and the third defines a null electromagnetic field.

For the consideration of the Pirani matrix we find that by proper choice of the tetrad $\lambda^i_{(a)}$ at a point the physical components of F_{ij} can be expressed in the following form for the three cases, viz.

$$\textit{Case (a)}: F_A = (\alpha, 0, 0, \beta, 0, 0) \qquad (9a)$$

$$\textit{Case (b)}: F_A = (0, 0, \alpha, 0, \beta, 0), \alpha \neq \beta \qquad (9b)$$

$$\textit{Case (c)}: F_A = (0, 0, \alpha, 0, \alpha, 0) \qquad (9c)$$

where F_A is the six-vector corresponding to the physical component $F_{(ab)}$:

$$F_{(ab)} = F_{ij}\lambda^i_{(a)}\lambda^j_{(b)}. \qquad (10)$$

The physical components of L_{hijk} with respect to this tetrad assume the form

$$L_{AB} = \begin{bmatrix} M & N \\ N & -M \end{bmatrix} \qquad (11)$$

in six-dimensional representation, M and N being symmetric matrices of third order. The matrices M and N in the three cases are given by

$$(a)\quad M = \text{diag}\left(\tfrac{1}{2}F_2,\ -\tfrac{1}{4}F_2,\ -\tfrac{1}{4}F_2\right),\ N = \text{diag}\left(2\alpha\beta,\ -\alpha\beta,\ -\alpha\beta\right) \quad .. \ (12a)$$

$$(b)\quad M = \text{diag}\left(-\tfrac{1}{4}F_2,\ \tfrac{1}{2}F_2-\tfrac{3}{2}\alpha^2,\ \tfrac{1}{2}F_2+\tfrac{3}{2}\beta^2\right),\ N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tfrac{3}{2}\alpha\beta \\ 0 & \tfrac{3}{2}\alpha\beta & 0 \end{bmatrix} \quad .. \ (12b)$$

$$(c)\quad M = \text{diag}\left(0,\ -\tfrac{3}{2}\alpha^2,\ \tfrac{3}{2}\alpha^2\right),\ N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tfrac{3}{2}\alpha^2 \\ 0 & \tfrac{3}{2}\alpha^2 & 0 \end{bmatrix} \quad .. \quad .. \quad .. \ (12c)$$

where

$$F_2 = F_{ij}F^{ij}. \quad .. \quad .. \quad .. \quad .. \ (13)$$

It is seen that the matrices (Pirani 1962) in the cases (a) and (b), i.e. for the non-null electromagnetic fields, are of type ID and for the case (c), i.e. for the null electromagnetic field, it is of type IIN.

Reference

Pirani, F. A. E. (1962). Recent Developments in General Relativity. Pergamon Press, London, pp. 89–105.