UNSTEADY FLOW OF VISCOELASTIC FLUIDS THROUGH CIRCULAR AND COAXIAL CIRCULAR DUCTS WITH PRESSURE GRADIENT AS ANY FUNCTION OF TIME

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In part A of this paper the problem when a linear viscoelastic Maxwell fluid flows through long circular ducts has been studied, the pressure gradient has been assumed to be any function of time. In part B some results of unsteady flow of linear viscoelastic Maxwell fluid through long coaxial circular ducts have been obtained, taking pressure gradient as any function of time. In both the cases a few particular cases of results with pressure gradients as any particular function of time, i.e. flow under an impulsive pressure gradient, flow under constant pressure gradient and flow under harmonically oscillating pressure gradient, have been discussed in detail. In both the parts results for ordinary viscous flow are deduced by making relaxation time and retardation time tend to zero. The deduced results are the same as obtained for ordinary viscous flow.

1. INTRODUCTION

Unsteady laminar viscous incompressible flow through large rectangular ducts has been studied by Chien Fan (1965) when the axial pressure gradient is an arbitrary function of time. Sharma (1970) has studied the same problem through large circular ducts and also through large coaxial circular ducts. Flow of viscoelastic Maxwell fluid through channels of different sections has been studied by various authors. Ghosh (1968) has studied the problem when a linear viscoelastic fluid flows through rectangular duct. He assumed the pressure gradient as any function of time.

In part A of this paper we present some results of unsteady flow of linear viscoelastic Maxwell fluid through long circular ducts. The pressure gradient has been assumed to be any function of time. In part B we have obtained some results of unsteady flow of linear viscoelastic Maxwell fluid through long coaxial circular ducts. A few particular cases of results with pressure gradients as any particular function of time have been discussed. In both the parts results for ordinary viscous flow are deduced by making relaxation time and retardation time tend to zero. The deduced results are the same as obtained by Sharma (1970) for ordinary viscous flow

2. Governing Equations

The rheological equations satisfied by viscoelastic liquids (Oldroyd 1958) are

$$\left.\begin{aligned} p_{ik} &= -p\delta_{ik}+p'_{ik} \\ p'_{ik}+\lambda_1\frac{D}{Dt}p'_{ik}+\mu_0 p'_{jj}e_{in}-\mu_1(p'_{ij}e_{jk}+p'_{jk}e_{ij})-\nu_1 p'_{jl}e_{jl}\delta_{ik} & \\ = 2\eta_0\left[e_{ik}+\lambda_2\frac{D}{Dt}e_{ik}-2\mu_2 e_{ij}e_{jk}+\nu_2 e_{jl}e_{jl}\delta_{ik}\right] & \end{aligned}\right\} \quad \ldots \quad (1)$$

with the equation of incompressibility

$$e_{ii} = 0$$

where

$$\frac{D}{Dt}b_{ik} = \frac{\partial}{\partial t}b_{ik}+v_j b_{ik,\,j}+w_{ij}b_{jk}+w_{kj}b_{ij}$$

$$e_{ik} = \tfrac{1}{2}(v_{k,\,i}+v_{i,\,k})$$

$$w_{ik} = \tfrac{1}{2}(v_{k,\,i}+v_{i,\,k})$$

and δ_{ik} is the Kronecker delta, e_{ik} the rate of strain tensor, p_{ik} the stress tensor, λ_1 the relaxation time, λ_2 the retardation time, η_0 the coefficient of viscosity and μ_0, μ_1, μ_2, ν_1 and ν_2 are material constants, each being of the dimension of time.

For

$$\eta_0 > 0,\ \lambda_1 = \mu_1 > \lambda_2 = \mu_2 \geqslant 0$$

$$\mu_0 = \nu_1 = \nu_2 = 0,$$

the liquid will exhibit Weissenberg climbing effect when sheared at a uniform rate between rotating cylinders.

For

$$\eta_0 > 0,\ \lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0$$

$$\mu_0 = \nu_1 = \nu_2 = 0,$$

the liquid will behave as ordinary viscous liquid.

PART A

3. Equation of Motion

The equation of motion in the absence of extraneous forces is

$$\rho\left[\frac{\partial v_i}{\partial t}+v_{i,\,j}v_j\right] = -p_{i,\,j}+p'_{ij,\,j} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where ρ is the density and p is the pressure.

We now consider a slow shearing motion through a circular duct with z-axis along the axis of the duct. Assuming

$$u = v = 0, \quad w = w(r, t)$$

the equation of motion is reduced to (Ghosh 1968)

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t} = -\frac{1}{\rho}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial z}+\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right) \quad .. \quad (3)$$

where

$$\nu = \frac{\eta_0}{\rho}.$$

Choosing $-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t)$, eqn. (3) becomes

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t} = \left(1+\lambda_1\frac{\partial}{\partial t}\right)f(t)+\nu\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right). \quad .. \quad (4)$$

The boundary conditions are

$$\left.\begin{array}{l} w = 0 \text{ when } r = a \\ w = 0 \text{ when } t \leq 0 \end{array}\right\}. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

4. Solution

Let us assume

$$w(r, t) = \sum_i A_i(t) J_0(r\xi_i) \quad .. \quad .. \quad .. \quad .. \quad (6)$$

where $A_i(t)$ is some function of time such that $A_i(t) = 0$ when $t = 0$ satisfying the boundary condition (5), that is $w = 0$ when $t = 0$, and ξ_i is the root of the equation

$$J_0(a\xi_i) = 0 \quad .. \quad .. \quad .. \quad .. \quad (7)$$

and the summation is to be taken on all values of ξ satisfying (7).

Substituting (6) in (7) and making use of the recurrence relation (Watson 1944)

$$J''_0(r\xi_i)+\frac{1}{r\xi_i}J'_0(r\xi_i)+J_0(r\xi_i) = 0$$

we easily get

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial A_i}{\partial t} = \{f(t)+\lambda_1 f'(t)\}\frac{2}{a\xi_i J_1(a\xi_1)}-\nu\xi_i^2\left(1+\lambda_2\frac{\partial}{\partial t}\right)A_i \quad .. \quad (8)$$

where ξ_i is the root of eqn. (7).

Applying Laplace's transforms to every step of (8), we get

$$\bar{A}_i(p)[\lambda_1 p^2+(1+\nu\lambda_2\xi_i^2)p+\nu\xi_i^2] = \frac{2}{a\xi_i J_1(a\xi_i)}[\bar{f}(p)+\lambda_1 p\bar{f}(p)]$$

$$\bar{A}_i(p) = \frac{2(1+\lambda_1 p)\bar{f}(p)}{a\xi_i J_1(a\xi_i)\{\lambda_1 p^2+(1+\nu\lambda_2\xi_i^2)p+\nu\xi_i^2\}}$$

$$= \frac{2\bar{f}(p)}{a\xi_i J_1(a\xi_i)\lambda_1(\alpha-\beta)}\left\{\frac{1+\lambda_1\alpha}{p-\alpha}-\frac{1+\lambda_1\beta}{p-\beta}\right\} \quad .. \quad .. \quad .. \quad (9)$$

where α, β are the roots of

$$\lambda_1 p^2+(1+\nu\lambda_2\xi_i^2)p+\nu\xi_i^2 = 0$$

i.e.

$$\alpha = \left[-(1+\nu\lambda_2\xi_i^2)+\sqrt{(1+\nu\lambda_2\xi_i^2)^2-4\lambda_1\nu\xi_i^2}\right]/(2\lambda_1)$$

and

$$\beta = \left[-(1+\nu\lambda_2\xi_i^2)-\sqrt{(1+\nu\lambda_2\xi_i^2)^2-4\lambda_1\nu\xi_i^2}\right]/(2\lambda_1)$$

whence

$$\lambda_1(\alpha-\beta) = \sqrt{(1+\nu\lambda_2\xi_i^2)^2-4\lambda_1\nu\xi_i^2}. \qquad (10)$$

Applying Convolution Theorem on (9), we get

$$A_i(t) = \frac{2}{a\xi_i J_1(a\xi_i)\lambda_1(\alpha-\beta)}\left[(1+\lambda_1\alpha)\int_0^t e^{\lambda\alpha}f(t-\lambda)\,d\lambda-(1+\lambda_1\beta)\int_0^t e^{\lambda\beta}f(t-\lambda)\,d\lambda\right].$$

Hence

$$w(r,t) = \frac{2}{a}\sum_i \frac{1}{\xi_i J_1(a\xi_i)\lambda_1(\alpha-\beta)}\left[(1+\lambda_1\alpha)\int_0^t e^{\lambda\alpha}f(t-\lambda)\,d\lambda-(1+\lambda_1\beta)\int_0^t e^{\lambda\beta}f(t-\lambda)\,d\lambda\right]J_0(r\xi_i) \qquad (11)$$

where ξ_i is the root of

$$J_0(a\xi_i) = 0. \qquad (12)$$

5. Flow under an Impulsive Pressure Gradient

Let

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) = A\,\delta(t)$$

where

$$\int_{-\infty}^{\infty} f(u)\delta(u-y)\,du = f(y).$$

Then from (11) we get

$$w(r,t) = \frac{2}{a}\sum_i \frac{1}{\xi_i J_1(a\xi_i)\lambda_1(\alpha-\beta)}\left[(1+\lambda_1\alpha)e^{\alpha t}-(1+\lambda_1\beta)e^{\beta t}\right]J_0(r\xi_i). \qquad (13)$$

Now if $\lambda_1 \to 0$, $\lambda_2 \to 0$, then by (10),

$$\lambda_1(\alpha-\beta) \to 1,$$

and also $\alpha \to -\nu\xi_i^2$, $\beta \to -\infty$.

Therefore, (13) becomes

$$w(r,t) = \frac{2}{a}\sum_i \frac{J_0(r\xi_i)}{\xi_i J_1(a\xi_i)}e^{-\nu\xi_i^2 t}. \qquad (14)$$

Expression (13) is the same as obtained by Sharma (1970) for an ordinary viscous flow when $A = 1$.

6. Flow under Constant Pressure Gradient

Let

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) = K_0\,H(t)$$

where K_0 is positive constant and $H(t)$ is the Heaviside unit step function.

Substituting it in (11) we get

$$w(r,\,t)=\frac{2}{a}\sum_{i}\frac{K_0J_0(r\xi_i)}{\xi_iJ_1a(\xi_i)\lambda_1(\alpha-\beta)}\left[\frac{e^{t\alpha}}{\alpha}-\frac{e^{t\beta}}{\beta}+\frac{\alpha-\beta}{\alpha\beta}+\lambda_1(e^{\alpha t}-e^{\beta t})\right]. \quad .. \quad (15)$$

Now if $\lambda_1\to 0$, $\lambda_2\to 0$

$$\lambda_1(\alpha-\beta)\to 1,\;\; \alpha\to -\nu\xi_i^2,\;\; \beta\to -\infty.$$

Therefore, (15) becomes

$$w(r,\,t)=\frac{2K_0}{a\nu}\sum_{i}\frac{J_0(r\xi_i)}{\xi_i^3J_1(a\xi_i)}\,(1-e^{-\nu\xi_i^2t}) \quad .. \quad .. \quad (16)$$

where ξ_i is the root of eqn. (12).

This is the same result as obtained by Sharma (1970) for an ordinary viscous fluid.

7. Flow under Harmonically Oscillating Pressure Gradient

Let

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}=f(t)=K\cos\omega t.$$

Then (11) becomes

$$w(r,\,t)=\frac{2K}{a}\sum_{i}\frac{J_0(r\xi_i)}{\xi_iJ_1(a\xi_i)\lambda_1(\alpha-\beta)}\left[(1+\lambda_1\alpha)\int_0^t e^{\lambda\alpha}\cos\omega(t-\lambda)\,d\lambda\right.$$
$$\left.-(1+\lambda_1\beta)\int_0^t e^{\lambda\beta}\cos\omega(t-\lambda)\,d\lambda\right]$$
$$=\frac{2K}{a}\sum_{i}\frac{J_0(r\xi_i)}{\xi_iJ_1(a\xi_i)\lambda_1(\alpha-\beta)}\left[\frac{(1+\lambda_1\alpha)(\alpha e^{t\alpha}-\alpha\cos\omega t+\omega\sin\omega t)}{\alpha^2+\omega^2}\right.$$
$$\left.-\frac{(1+\lambda_1\beta)(\beta e^{t\beta}-\beta\cos\omega t+\omega\sin\omega t)}{\beta^2+\omega^2}\right]. \quad .. \quad .. \quad .. \quad .. \quad (17)$$

When $\lambda_1\to 0$, $\lambda_2\to 0$

$$\lambda_1(\alpha-\beta)\to 1,\;\; \alpha\to -\nu\xi_i^2,\;\; \beta\to -\infty.$$

Therefore, (17) becomes

$$w(r,\,t)=\frac{2K}{a}\sum_{i}\frac{J_0(r\xi_i)}{\xi_iJ_1(a\xi_i)(\omega^2+\nu^2\xi_i^4)}\left[\nu\xi_i^2\cos\omega t+\omega\sin\omega t-\nu\xi_i^2e^{\nu\xi_i^2t}\right] \quad (18)$$

where ξ_i is the root of eqn. (12).

This is the same result as obtained by Sharma (1970) for an ordinary viscous fluid.

Discussion

If the value of i is such that the inequality

$$\left[(2\lambda_1-\lambda_2)-2\sqrt{\lambda_1(\lambda_1-\lambda_2)}\right]<\nu\lambda_2^2\xi_i^2<\left[(2\lambda_1-\lambda_2)+2\sqrt{\lambda_1(\lambda_1+\lambda_2)}\right]$$

is satisfied, the values of α and β are complex quantities. Therefore, in this case the expression for w will contain sine and cosine terms. This shows that, unlike the ordinary viscous fluid, the non-steady flow of viscoelastic liquid possesses periodic vibrations whose amplitude decreases with time.

PART B

Now we shall consider the unsteady flow of viscoelastic fluids through large coaxial circular ducts. Let a, b be the radii of the cross-section where $a > b$.

8. Flow under an Impulsive Pressure Gradient

Equation (3) still holds but we have to integrate it under the following boundary conditions:

$$\left.\begin{aligned} w &= 0 \text{ when } r = a \\ w &= 0 \text{ when } r = b \\ w &= 0 \text{ when } t \leq 0 \end{aligned}\right\}. \qquad (19)$$

Let us assume

$$w(r,\, t) = \sum_i A_i(t)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\} \qquad (20)$$

where $A_i(t)$ is some function of time such that $A_i(t) = 0$ when $t = 0$, satisfying the boundary condition (19), i.e. $w = 0$ when $t = 0$, and ξ_i is the root of the equation

$$J_0(b\xi_i)Y_0a(\xi_i) - J_0(a\xi_i)Y_0(b\xi_i) = 0.$$

Proceeding as in part A, eqn. (8) becomes

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial A_i}{\partial t} = \{f(t)+\lambda_1 f'(t)\}\frac{2J_0(b\xi_i)}{J_0(a\xi_i)+J_0(b\xi_i)} - \nu\xi_i^2\left(1+\lambda_2\frac{\partial}{\partial t}\right)A_i. \qquad (21)$$

Applying Laplace's transform to every step of (21), we get

$$\bar{A}_i(p) = \frac{2J_0(b\xi_i)\bar{f}(p)}{\{J_0(a\xi_i)+J_0(b\xi_i)\}\lambda_1(\alpha-\beta)}\left[\frac{1+\lambda_1\alpha}{p-\alpha} - \frac{1+\lambda_1\beta}{p-\beta}\right] \qquad (22)$$

which corresponds to eqn. (9) of part A and α, β are again the roots of

$$\lambda_1 p^2 + (1+\nu\lambda_2\xi_i^2)p + \nu\xi_i^2 = 0$$

whence if $\lambda_1 \to 0$, $\lambda_2 \to 0$,

$$\lambda_1(\alpha-\beta) \to 1, \quad \alpha \to -\nu\xi_i^2, \quad \beta_2 \to -\infty. \qquad (23)$$

Applying Convolution Theorem on (22), we get

$$A_i(t) = \sum_i \frac{2J_0(b\xi_i)}{\{J_0(a\xi_i)+J_0(b\xi_i)\}\lambda_1(\alpha-\beta)}\left[(1+\lambda_1\alpha)\int_0^t e^{\lambda\alpha}f(t-\lambda)\,d\lambda - (1+\lambda_1\beta)\int_0^t e^{\lambda\beta}f(t-\lambda).\right. \qquad (24)$$

Then eqn. (20) becomes

$$w(r,\, t) = \sum_i \frac{2J_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)}{\{J_0(a\xi_i)+J_0(b\xi_i)\}\lambda_1(\alpha-\beta)}\Big[(1+\gamma_1\alpha)$$
$$\times \int_0^t e^{\lambda\alpha}f(t-\lambda)\,d\lambda - (1+\lambda_1\beta)\int_0^t e^{\lambda\beta}f(t-\lambda)\,d\lambda\Big] \qquad (25)$$

where ξ_i is the root of the equation

$$J_0(b\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(b\xi_i) = 0. \quad \ldots \quad (26)$$

9. Flow under an Impulsive Pressure Gradient

Let

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) = A\,\delta(t).$$

Then from (25) we get

$$w(r, t) = \sum_i \frac{2J_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\}}{\{J_0(a\xi_i) + J_0(b\xi_i)\}\lambda_1(\alpha-\beta)}\{e^{\alpha t} - e^{\beta t} + \lambda_1\alpha e^{\alpha t} - \lambda_1\beta e^{\beta t}\} \quad (27)$$

which corresponds to eqn. (13) of part A.

When $\lambda_1 \to 0$, $\lambda_2 \to 0$, eqn. (27) becomes

$$w(r, t) = \sum_i \frac{2J_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\}}{J_0(a\xi_i) + J_0(b\xi_i)} e^{-\nu\xi_i^2 t} \quad \ldots \quad (28)$$

where (28) is the same as obtained by Sharma (1970) for an ordinary viscous flow when $A = 1$.

10. Flow under Constant Pressure Gradient

Let

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t) = K_0\,H(t).$$

Then on substitution and simplifying a little, eqn. (25) becomes

$$w(r, t) = \sum_i \frac{2K_0J_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\}}{\{J_0(a\xi_i) + J_0(b\xi_i)\}\lambda_1(\alpha-\beta)}\left\{\frac{e^{\alpha t}}{\alpha} - \frac{e^{\beta t}}{\beta} + \frac{\alpha-\beta}{\alpha\beta} + \lambda_1(e^{\alpha t} - e^{\beta t})\right\}. \quad \ldots \quad (29)$$

Now if $\lambda_1 \to 0$, $\lambda_2 \to 0$, eqn. (29) reduces to

$$w(r, t) = \sum_i \frac{2K_0J_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\}}{\nu\xi_i^2\{J_0(a\xi_i) + J_0(b\xi_i)\}}(1 - e^{-\nu\xi_i^2 t}) \quad \ldots \quad (30)$$

where ξ_i is the root of (26).

The expression (30) is the same as obtained by Sharma (1970) for any ordinary viscous liquid.

11. Flow under Harmonically Oscillating Pressure Gradient

Let

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = f(t)\,K\cos\omega t.$$

Then eqn. (25) reduces on simplification to

$$w(r, t) = \sum_i \frac{2KJ_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\}}{J_0(a\xi_i) + J_0(b\xi_i)}\left\{\frac{(1+\lambda_1\alpha)(\alpha e^{t\alpha} - \alpha\cos\omega t + \omega\sin\omega t)}{\alpha^2+\omega^2} - \frac{(1+\lambda_1\beta)(\beta e^{t\beta} - \beta\cos\omega t + \omega\sin\omega t)}{\beta^2+\omega^2}\right\}. \quad \ldots \quad (31)$$

When $\lambda_1 \to 0$, $\lambda_2 \to 0$, eqn. (31) becomes

$$w(r, t) = \sum_i \frac{2KJ_0(b\xi_i)\{J_0(r\xi_i)Y_0(a\xi_i) - J_0(a\xi_i)Y_0(r\xi_i)\}}{J_0(a\xi_i)+J_0(b\xi_i)} \cdot \frac{1}{\alpha^2+\omega^2}$$

$$\times\{\nu\xi_i^2 \cos \omega t + \omega \sin \omega t - \nu\xi_i^2 e^{-\nu\xi_i^2 t}\} \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where ξ_i is the root of eqn. (26).

This is the same result as obtained by Sharma (1970) for an ordinary viscous liquid.

It can be verified that when $b \to 0$, all the results of part B reduce to the corresponding results of part A.

As discussed earlier in part A non-steady flow of viscoelastic liquid possesses periodic vibrations whose amplitude decreases with time.

References

Chien Fan (1965). Unsteady laminar incompressible flow through rectangular ducts with pressure gradient as any function of time. *Z. angew. Math. Phys.*, **16**, 356–60.

Ghosh, A. K. (1968). Note on the flow of viscoelastic fluids through rectangular ducts with pressure gradient as any function of time. *Bull. Calcutta math. Soc.*, **60**, 163–68.

Oldroyd, J. G. (1958). Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. R. Soc.*, A **245**, 278–97.

Sharma, H. S. (1970). Unsteady laminar incompressible flow through circular and coaxial circular ducts with pressure gradient as any function of time. *Agra Univ. Res. J.*, **XIX**, Part II, pp. 1–4.

Watson, G. M. (1944). Theory of Bessel Functions. Cambridge University Press.