THERMO-ELASTIC-PLASTIC TRANSITION OF TUBES UNDER UNIFORM PRESSURE AND STEADY STATE TEMPERATURE*

by BOLINDRA N. BORAH,† *Department of Mathematics, North Carolina A. and T. State University, U.S.A.*

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Seth's elastic-plastic transition theory, which requires neither the yield criteria nor the associated flow rule, has been used to derive the transitional and plastic stresses and strains in tubes under uniform pressure and steady state temperature. It has been shown that some of the results obtained here agree with observations of Darrington, Johnson and Bland and found to agree with the well-known maximum shear theory. It has also been observed that constitutive equations for transition and plastic states grow from the results and take the Levy-Mises form.

1. INTRODUCTION

The problems involving thermo-elastic-plastic transition are investigated in current literature by employing method of superposition and using semi-empirical laws. The use of such methods is possible only in the linear theory where the effects are superposable. But in practice such effects are non-linear in character and, therefore, the method of superposition is not applicable. Again, the problem is solved first for the elastic range and later the solution for the plastic range is obtained by using the so-called yield condition. The perfect elasticity and ideal plasticity are two extreme properties of material. The use of such *ad hoc* rules like yield conditions amounts to divide the two extreme properties by a sharp line, which is physically not possible.

The transition theory developed by Seth provides a straightforward answer to the difficulties mentioned above. He has solved several problems of practical importance applying his new technique (Seth 1963, 1964) and results were compared with classical ones. The author has used this technique to obtain the thermo-elastic-plastic stresses and strains in shells (Borah 1970).

In this paper we study the problem of thermo-elastic-plastic transition in finite tubes using Seth's transition theory. The stresses and strains are obtained both in transition and plastic states without resorting to any *ad hoc* semi-empirical laws and associated flow rules. Our results obtained here

* Research partly supported by a grant of the Atomic Energy Commission to the Department of Mathematics, Oregon State University.

† *Present address:* 2706 Dumont Drive, Greensboro, N.C. 27403, U.S.A.

compare with some of the experimental observation studied by Johnson and Derrington (1958) and Wilhoit (1958). The constitutive equations for both transition and fully plastic states are also derived from the results and they take the Levy-Mises form.

2. Formulation of the Problem and Identification of the Transition Points

We consider here a tube of internal and external radii a and b $(a < b)$ respectively subjected to uniform pressure p and steady state temperature θ on the inner surface $r = a$. Further, if we assume that there are no body forces, body couples and couple stresses on the tube, and if only a steady deformation problem is considered, then the basic equations characterizing the problem may be written in the cylindrical coordinate system as follows:

$$\tau_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2\mu e_{ij} - \omega\theta\delta_{ij} \quad \ldots \quad (2.1)$$

$$K\theta_{,ii} = 0 \quad \ldots \quad (2.2)$$

$$\tau_{ij,j} = 0 \quad \ldots \quad (2.3)$$

$$\tau_{ij} = \tau_{ji} \quad \ldots \quad (2.4)$$

where τ_{ij}, e_{ij} are stress and strain tensors, δ_{ij} is Kronecker delta, θ is rise in temperature, λ, μ are known as Lame's constant and $\omega = \alpha(3\lambda + 2\mu)$, α being the coefficient of thermal expansion. The comma represents differentiation with respect to the radius r. The eqn. (2.1) is constitutive equation, known as modified Hooke's Law; (2.2) is heat conduction equation in the most simplified form; (2.3) represents equation of equilibrium; and (2.4) states that the stress tensor τ_{ij} is symmetric.

On account of axial symmetry of the problem, the displacement field may be chosen as

$$u = r(1-\beta), \quad v = 0, \quad w = z(1-d_0),$$

where $\beta = f(r)$ and $r = (x^2+y^2)^{\frac{1}{2}}$, d_0 being a constant to be determined, $(d_0 < 1)$. By using the Almansi Strain measure we have the following expressions for strains:

$$\left.\begin{aligned} e_{rr} &= \tfrac{1}{2}[1-(r\beta'+\beta)^2] \\ e_{\theta\theta} &= \tfrac{1}{2}[1-\beta^2], \quad e_{zz} = \tfrac{1}{2}[1-d_0^2] \\ e_{r\theta}, &\quad e_{\theta z}, \quad e_{rz} = 0 \\ \text{and} \quad e_{\alpha\alpha} &= 1-\tfrac{1}{2}(r\beta'+\beta)^2 - \tfrac{1}{2}\beta^2 + \tfrac{1}{2}(1-d_0^2) \end{aligned}\right\} \quad \ldots \quad (2.5)$$

the prime indicates the differentiation with respect to r. The stresses obtained from (2.1) and (2.5) are

$$\left.\begin{aligned} \tau_{rr} &= \lambda[1-\tfrac{1}{2}(r\beta'+\beta)^2-\tfrac{1}{2}\beta^2+\tfrac{1}{2}(1-d_0^2)]+\mu[1-(r\beta'+\beta)^2]-\omega\theta \\ \tau_{\theta\theta} &= \lambda[1-\tfrac{1}{2}(r\beta'+\beta)^2-\tfrac{1}{2}\beta^2+\tfrac{1}{2}(1-d_0^2)]+\mu(1-\beta^2)-\omega\theta \\ \tau_{zz} &= \lambda[1-\tfrac{1}{2}(r\beta'+\beta)^2-\tfrac{1}{2}\beta^2+\tfrac{1}{2}(1-d_0^2)]+\mu(1-d_0^2)-\omega\theta \\ \text{and} \quad & \tau_{zr}, \quad \tau_{r\theta}, \quad \tau_{\theta z} = 0. \end{aligned}\right\} \quad \ldots \quad (2.6)$$

The temperature field given by (2.2) is

$$\theta = \frac{\theta_0 \log (b/r)}{\log (b/a)}$$

where

$$\theta = \begin{cases} 0 & \text{for } r = b \\ \theta_0 & \text{for } r = a. \end{cases}$$

The only equilibrium equation which remains to be satisfied is

$$\frac{d\tau_{rr}}{dr} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0. \qquad \text{.. .. .. ..} \quad (2.7)$$

The other two equations are satisfied identically.

Substituting (2.6) in (2.7) we have

$$\beta^2 + (r\beta' + \beta)^2 - K_0 \log r + c\int r\beta'^2 \, dr = K \qquad \text{.. ..} \quad (2.8)$$

where

$$K_0 = \frac{2\omega\theta_0}{(\lambda+2\mu)\log b/a}, \quad c = \frac{2\mu}{\lambda+\mu}$$

and K is a constant of integration. Setting $\log r = z$ in (2.8) and then differentiating with respect to z, we get

$$2\beta\beta' + 2(\beta' + \beta'')(\beta' + \beta) - K_0 + c\beta'^2 = 0. \qquad \text{.. ..} \quad (2.9)$$

Setting again $\beta = \sqrt{K_0}\, p$ and $p + p' = q$, the equation (2.9) reduces to

$$2(q-p)p + 2qq' + c(q-p)^2 = 1$$

which may also be put in the form

$$\frac{dQ}{dF} = \frac{QF(F-1)}{F^3 - \left(1 - \frac{c}{2}\right)F^2 - (c-1)F - \left(\frac{Q^2}{2} - \frac{c}{2} + 1\right)} \qquad \text{.. ..} \quad (2.10)$$

where $\frac{q}{p} = F, \frac{1}{p} = Q$. We assume that β has all necessary properties and does not vanish throughout the elastic domain. From (2.10) it may be observed that some of the possible transition points of the differential system of the problem under consideration are

$$F = 0, \ F = 1 \text{ and } F = \pm\infty, \text{ where } F = \left(1 - \frac{\beta' r}{\beta}\right).$$

There may be other transition points given by the roots of the expression in the denominator of (2.10) which may represent various physical phenomena.

In order to study further about the nature of those points, we consider the reciprocal deformation ellipsoid

$$(1-2e_{rr})\, dr^2 + (1-2e_{\theta\theta})r^2 \, d\theta^2 + (1-2e_{zz})\, dz^2 = K^2$$

which may be rewritten with the help of (2.5) as

$$\beta^2 F \, dr^2 + \beta^2 r^2 \, d\theta^2 + d_0{}^2 \, dz^2 = K^2 \qquad \text{.. .. ..} \quad (2.11)$$

where K is some constant. It may be seen now $F = 0$ and $F = \pm\infty$ are transition points, for the first case the ellipsoid becomes a cylinder and for the second case the ellipsoid tends to become a pair of plans. The third point $F = 1$ is a regular point because, then the ellipsoid (2.11) becomes

$$\beta^2\, dr^2 + \beta^2 r^2\, d\theta^2 + d_0^2\, dz^2 = K^2$$

where β becomes some constant. It can be very easily seen that $F = 0$ corresponds to infinite extension and $F = \pm\infty$ corresponds to infinite contraction.

Now we observe that the material from elastic state can go over into (1) plastic state, or (2) to creep state, or (3) first to plastic and then to creep or vice versa under external loading. All these final states are reached through a transition state. Therefore, the transition can take place either through the principal stresses τ_{rr}, $\tau_{\theta\theta}$ or τ_{zz} becoming critical or through the principal stress differences $\tau_{rr}-\tau_{\theta\theta}$, $\tau_{\theta\theta}-\tau_{zz}$ or $\tau_{zz}-\tau_{rr}$ becoming critical. But it will be seen later that transitional values of the stresses obtained through $\tau_{\theta\theta}$ and τ_{zz} remain analytically indetermined, because of the presence of two constants of integration and one available boundary condition. Hence we have to consider only the following two cases:

(*a*) Transition through τ_{rr}.

(*b*) Transition through $\tau_{rr}-\tau_{\theta\theta}$.

For each transition points $F = 0$, $F = \pm\infty$ we have to determine the stresses and strains corresponding to the above two cases.

3. Determination of Stresses and Strains in the Transition and Plastic States

We shall now determine the stresses and strains in the transition and plastic states corresponding to the transition points mentioned in section 2.

Case I: $F \to \pm\infty$—*Infinite Contraction*

In this case the tube is subjected to uniform pressure and steady state temperature on the inner surface of the tube.

(a) *Transition through* τ_{rr}—The first equation in (2.6) may be rewritten as

$$R = \beta^2[(1-c)+F^2] \quad \ldots \quad (3.1)$$

where

$$R \equiv -\frac{c}{\mu}\tau_{rr} + D_0 - \frac{c}{\mu}\beta_0 \log\frac{b}{r}$$

$$D_0 = (3-2c)-(1-c)d_0^2$$

$$\beta_0 = \frac{\omega\theta_0}{\log\dfrac{b}{a}}$$

and

$$c = \frac{2\mu}{\lambda+2\mu}.$$

Taking logarithmic differentiation with respect to r we have from (3.1) and (2.10)

$$\frac{d(\log R)}{d(\log r)} = -\frac{cF^2+c+Q^2}{(1-c)+F^2}.$$

Hence $R = A_0 r^{-c}$ as $F \to \pm\infty$, where A_0 is some arbitrary constant. The boundary condition $\tau_{rr} = 0$ when $r = b$ and $R = A_0 r^{-c}$ give

$$\tau_{rr} = \frac{\mu}{c} D_0 \left[1-\left(\frac{b}{r}\right)^c\right] - \beta_0 \log \frac{b}{r}. \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

Then from the equilibrium eqn. (2.7) we have

$$\tau_{\theta\theta}-\tau_{rr} = D_0\mu\left(\frac{b}{r}\right)^c + \beta_0. \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

Applying the same technique as above to $(\tau_{\theta\theta}-\tau_{zz})$ and $(\tau_{rr}-\tau_{zz})$ we obtain

$$\tau_{\theta\theta}-\tau_{zz} = \mu d_0^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

and

$$\tau_{rr}-\tau_{zz} = \mu d_0^2 - \mu D_0\left(\frac{b}{r}\right)^c - \beta_0. \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

The initiation of yield starts at $r = a$ and the pressure for which the material starts to yeild may be obtained from (3.2) as

$$p_i = \frac{D_0\mu}{c}\left[\left(\frac{b}{a}\right)^2 - 1\right] + \beta_0 \log (b/a).$$

It may be seen (Seth 1964) that the pressure for the similar case without the presence of temperature may be obtained as

$$p_i^* = \frac{D_0\mu}{c}\left[\left(\frac{b}{a}\right)^c - 1\right].$$

Hence it is clear that outward flow of heat opposes the yielding.

In order to obtain d_0, if F is the force applied at the ends of the cylinder, then $F = 2\pi \int_a^b \tau_{zz} r\, dr$, therefore, d_0 can be obtained. If $d_0 = 1$ then $e_{zz} = 0$, which is the case of plane strain.

In fully plastic state, $c \to 0$ and from (3.2), (3.3), (3.4) and (3.5) we obtain

$$\left.\begin{aligned} \tau_{rr} &= \tfrac{2}{3}Y(3-d_0^2)+\beta_0^* \log r/b \\ \tau_{\theta\theta}-\tau_{rr} &= \tfrac{2}{3}Y(3-d_0^2)+\beta_0^* \\ \tau_{\theta\theta}-\tau_{zz} &= \tfrac{2}{3}Yd_0^2 \\ \tau_{zz}-\tau_{rr} &= 2Y+\beta_0^*-\tfrac{4}{3}Yd_0^2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

where Y is yield stress in tension. None of the above result in (3.6) is independent of d_0, therefore no relation may be used as yield condition; however,

$$\tau^d_{\theta\theta} = \tfrac{2}{3}Y + \frac{\beta_0{}^*}{3}$$

where $\tau^d_{\theta\theta}$ is called deviatoric stress; is seen to be independent of d_0 and, therefore, may be used as yield condition,

$$\left[\beta_0{}^* = \lim_{c \to 0} \frac{\alpha(3\lambda+2\mu)\theta_0}{\log \frac{b}{a}}\right].$$

We may add here that τ_{zz} is seen to be the intermediate stress ($\tau_{rr} \leqslant \tau_{zz} \leqslant \tau_{\theta\theta}$) both in elastic and plastic state; but generally it is assumed to be so in current literature (Hill 1950, Bland 1956).

Considering the case of plane strain ($d_0 = 1$) we may see from (3.6) that maximum stress-difference is

$$\tau_{\theta\theta} - \tau_{rr} = \tfrac{4}{3}Y + \beta_0{}^*.$$

Writing in full we have

$$\tau_{\theta\theta} - \tau_{rr} = \tfrac{4}{3}Y + \frac{\omega_0\theta_0}{\log \frac{b}{a}}, \quad \omega_0 = \lim_{c \to 0} \alpha(3\lambda+2\mu). \qquad (3.7)$$

The same 'stress-saving' phenomenon of Johnson and Derrington (1958) which has been observed in shells (Borah 1970) may be also seen to occur from (3.7).

(b) *Transition through* ($\tau_{\theta\theta} - \tau_{zz}$)—We may write from first and second equations of (2.6) as

$$R \equiv \frac{\tau_{rr} - \tau_{\theta\theta}}{\mu} = \beta^2 . [1 - F^2]$$

and obtain as in (*a*)

$$\tau_{rr} - \tau_{\theta\theta} = \mu A_0 r^{-c} \text{ as } F \to \pm\infty \qquad (3.8)$$

where A_0 is an arbitrary constant. The equilibrium equation and boundary condition yield

$$\tau_{rr} = \frac{\mu B_0}{c}\left[1 - \left(\frac{b}{r}\right)^c\right] \qquad (3.9)$$

where $B_0 = -b^{-c}A_0$ being a parameter. Therefore (3.8) may be rewritten as

$$\tau_{\theta\theta} - \tau_{rr} = \mu B_0 \left(\frac{b}{r}\right)^c. \qquad (3.10)$$

The parameter B_0 may be obtained from (3.9) using the condition

$$(\tau_{rr})_{r=a} = -p_i$$

Therefore

$$B_0 = \frac{cp_i}{\mu\left[\left(\frac{b}{a}\right)^c - 1\right]}.$$

A similar treatment as before gives

$$(\tau_{\theta\theta}-\tau_{zz}) = \mu d_0^2 \quad \ldots \quad (3.11)$$

$$(\tau_{rr}-\tau_{zz}) = \mu d_0^2-\mu B_0\left(\frac{b}{r}\right)^c. \quad \ldots \quad (3.12)$$

Now we observe by comparison that $|\tau_{\theta\theta}-\tau_{rr}|$ remains as the greatest in value among all the stresses or their differences in both (*a*) and (*b*). Hence transition occurs through $\tau_{\theta\theta}-\tau_{rr}$ in the problem under consideration and thus agrees with the maximum shear theory of classical elasticity.

Case II: $F \to 0$—*Infinite Extension*

In this case the tube is subjected to an uniform pressure on its external surface and steady state temperature on its interior surface.

In the rest of this section, we shall only state the results obtained in (*a*) and (*b*). The same may be varified in like manner as in Case I.

(a) *Transition through* τ_{rr}—Here in the fully transition state we obtain the following results:

$$\left.\begin{aligned}\tau_{rr} &= \frac{\mu}{c}\left[D_0-\left(D_0-\frac{c\omega\theta_0}{\mu}\right)\left(\frac{r}{a}\right)^{\frac{c}{1-c}}\right]+\beta_0 \log\frac{r}{b}\\ \tau_{\theta\theta}-\tau_{rr} &= \left(\frac{c\omega\theta_0-\mu D_0}{1-c}\right)\left(\frac{r}{a}\right)^{\frac{c}{1-c}}+\beta_0\\ \tau_{zz}-\tau_{rr} &= \mu d_0^2\\ \tau_{\theta\theta}-\tau_{zz} &= \mu d_0^2\end{aligned}\right\} \quad \ldots \quad (3.13)$$

In the fully plastic state $c \to 0$ and we have

$$\left.\begin{aligned}\tau_{rr} &= \omega_0\theta_0-\tfrac{2}{3}(3-d_0^2)Y\log\frac{r}{a}-\beta_0^*\log\frac{b}{r}\\ \tau_{\theta\theta}-\tau_{rr} &= \tfrac{2}{3}Y(3-d_0^2)-\beta_0^*\\ \tau_{zz}-\tau_{rr} &= \tfrac{2}{3}Yd_0^2\\ \tau_{\theta\theta}-\tau_{zz} &= \tfrac{2}{3}Yd_0^2\end{aligned}\right\} \quad \ldots \quad (3.14)$$

None of the above relations in (3.14) is independent of d_0, however,

$$\tau_{rr}^d = \tfrac{4}{3}Y-\beta_0^*/3$$

which may be used as yield condition.

(b) *Transition through* $(\tau_{\theta\theta}-\tau_{rr})$—In this case the stress in the fully transition state are the following:

$$\tau_{rr} = \frac{\mu A_0}{3-c}\left[1-\left(\frac{a}{r}\right)^{4-c}\right] \quad \ldots \quad (3.15)$$

where

$$A_0 = \frac{+(3-c)p_0}{\mu\left[1-\left(\frac{a}{b}\right)^{4-c}\right]},$$

p_0 is the pressure applied on the exterior surface of the tube.

$$\tau_{\theta\theta}-\tau_{rr} = \mu A_0\left(\frac{a}{r}\right)^{4-c} \quad \ldots \quad (3.16)$$

$$\tau_{\theta\theta}-\tau_{zz} = \mu d_0^2 \quad \ldots \quad (3.17)$$

$$\tau_{zz}-\tau_{rr} = \mu d_0^2. \quad \ldots \quad (3.18)$$

The stresses for the fully plastic state may similarly be obtained making $c \to 0$.

Again comparing the results in (*a*) and (*b*) we come to conclusion that transition occurs through $|\tau_{\theta\theta}-\tau_{rr}|$.

Now we determine the strains in the transition states, the strains in the plastic state may be obtained letting $c \to 0$.

For $F \to \pm\infty$

The asymptotic values of the strains are calculated by following the same technique as employed in the other cases. Hence from (2.5) we have

$$\left.\begin{aligned} e_{rr} &= \tfrac{1}{2}\left[1-E\left(\frac{b}{r}\right)^{c}\right] \\ e_{\theta\theta} &= \tfrac{1}{2} \\ e_{zz} &= \tfrac{1}{2}[1-d_0^2] \\ e_{\alpha\alpha} &= \tfrac{1}{2}\left[3-d_0^2-E\left(\frac{b}{r}\right)^{c}\right] \end{aligned}\right\} \quad \ldots \quad (3.19)$$

where E is some arbitrary parameter.

For $F \to 0$

The asymptotic values of the strains in this case are the following (from (2.5)):

$$\left.\begin{aligned} e_{rr} &= \tfrac{1}{2} \\ e_{\theta\theta} &= \tfrac{1}{2}\left[1-G\left(\frac{a}{r}\right)^{c}\right] \\ e_{zz} &= \tfrac{1}{2}[1-d_0^2] \\ e_{\alpha\alpha} &= \tfrac{1}{2}\left[3-d_0^2-G\left(\frac{a}{r}\right)^{c}\right] \end{aligned}\right\} \quad \ldots \quad (3.20)$$

where G is some parameter.

4. The Constitutive Equations for the Transition and Plastic State

We shall now obtain the constitutive equations for transition and plastic states corresponding to the cases $F \to 0$ and $F \to \pm\infty$. As we have already observed in the preceeding sections, that transition occurs through $\tau_{\theta\theta}-\tau_{rr}$ in both the cases, we only obtain the constitutive equations using the transitional results obtained through $|\tau_{\theta\theta}-\tau_{rr}|$.*

For $F\to\pm\infty$

From (3.9), (3.10), (3.11), (3.12) and (3.19) we have the following:

$$\left.\begin{aligned} e^d_{rr} &= \frac{1}{2\mu}\tau^d_{rr} \\ e^d_{\theta\theta} &= \frac{1}{2\mu}\tau^d_{\theta\theta} \\ e^d_{zz} &= \frac{1}{2\mu}\tau^d_{zz} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

(where e^d_{rr}, τ^d_{rr}, etc., are deviatoric strains and stresses) provided we choose $E=B_0$. Therefore, the arbitrary constant E obtained in connection with transitional strains and the arbitrary constant B_0 obtained in connection with transitional stresses must be equal.

Therefore, (4.1) leads to the following

$$e^d_{ii} = \frac{1}{2\mu}\tau^d_{ii} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

and the parameter

$$E = \frac{cp_i}{\mu\left[\left(\frac{b}{a}\right)^c - 1\right]}.$$

For the fully plastic state the constitutive equation may be obtained from (4.2) as

$$\dot{e}^d_{ii} = \frac{3}{4Y}\tau^d_{ii} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

For $F\to 0$

From (3.15), (3.16), (3.17), (3.18) and (3.20) we obtain the following

$$e^d_{ii} = \frac{1}{2\mu}\tau^d_{ii} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

provided again we choose $G=A_0$. Therefore

$$G = \frac{-(3-c)p_0}{\mu\left[1-\left(\frac{a}{b}\right)^{4-c}\right]}.$$

The constitutive equation for the plastic state may be obtained from (4.4) as

$$\dot{e}^d_{ii} = \frac{3}{4Y}\tau^d_{ii}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

* However, the same constitutive equation may be obtained if we use the transitional values of strain and stress obtained through τ_{rr}.

Now we observe that the constitutive equations for the fully incompressible states ($c \to 0$) for both cases ($F \to \pm\infty$, $F \to 0$) given by equations (4.3) and (4.5) are the same equations. Therefore, they must represent the same state which is the plastic state in this problem.

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