AN ASSOCIATED GENERALIZED HERMITE POLYNOMIAL

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(Communicated by R. S. Mishra, F.N.A.)

(*Received 6 June 1970*)

In this paper, the author defined a polynomial associated with the generalized Hermite polynomial defined by Lahiri (1971) earlier. Hypergeometric form, generating relations, nth differential formulae and the differential equation for associated generalized Hermite polynomial have been derived in this paper.

1. Introduction

In a previous paper Lahiri (1971) has defined the generalized Hermite polynomial $H_{n,m,\nu}(x)$ as

$$H_{n,m,\nu}(x) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k \, n\,! \, (\nu x)^{n-mk}}{k\,! \; (n-mk)\,!} \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

where m is a positive integer and n is a non-negative integer.

We have shown (Lahiri 1971) that

$$\text{(i)} \;\; H_{n,m,\nu}(x) = (\nu x)^n \, {}_mF_0\left[\Delta(m\,;\; -n)\,;\; -\,;\; -\left(-\frac{m}{\nu x}\right)^m\right] \quad \ldots \quad \ldots \quad (1.2)$$

for $m = \nu = 2$, $H_{n,m,\nu}(x)$ reduces to the Hermite polynomial $H_n(x)$.

$$\text{(ii)} \;\; H_{n,m,\nu}(x) = (-1)^n \nu^{(1-m)n} D^{(m-1)n} \; {}_1F_{m-1}\left[-n\,;\; (m(1)\,;\; 1)\,;\; \left(-\frac{\nu x}{m}\right)^m\right] \quad (1.3)$$

$$\text{(iii)} \;\; H_{n,m,\nu}(x) = \frac{\nu^n n\,!}{(mn)\,!} D^{(m-1)n}\left[x^{mn} \, {}_mF_0\left\{\Delta(m\,;\; -mn)\,;\; -\,;\; -\left(-\frac{m}{\nu x}\right)^m\right\}\right] \quad \ldots \quad (1.4)$$

Lahiri (1969–70) has also shown that the differential equation satisfied by $H_{n,m,\nu}(x)$ as

$$m\frac{d^m}{dx^m} H_{n,m,\nu}(x) - \nu^m x \frac{d}{dx} H_{n,m,\nu}(x) + n\nu^m H_{n,m,\nu}(x) = 0 \quad \ldots \quad (1.5)$$

where $n > m-1$.

We shall be using the following notations:

$$\text{(i)} \quad \Delta(b\,;\,a) = \frac{a}{b}, \frac{a+1}{b}, \ldots, \frac{a+b-1}{b}$$

$$\text{(ii)} \quad \Delta_k(b\,;\,a) = \left(\frac{a}{b}\right)_k \left(\frac{a+1}{b}\right)_k \cdots \left(\frac{a+b-1}{b}\right)_k$$

$$\text{(iii)} \quad \Delta[a(1)\,;\,b] = \frac{b}{a}, \frac{b+1}{a}, \ldots, \frac{b+a-2}{a}$$

i.e. the last one term of the sequence $\Delta(a;\,b)$ is deleted.

$$\text{(iv)} \quad \Delta_k[a(1)\,;\,b] = \left(\frac{b}{a}\right)_k \left(\frac{b+1}{a}\right)_k \cdots \left(\frac{b+a-2}{a}\right)_k.$$

2. Definition and Hypergeometric Form

From (1.1), we have

$$\frac{d^p}{dx^p}[H_{n,m,\nu}(x)] = \sum_{k=0}^{\left[\frac{n-p}{m}\right]} \frac{(-1)^k\, n!\, \nu^{n-mk} x^{n-p-mk}}{k!\,(n-p-mk)!}.$$

Let us define the associated generalized Hermite polynomial as:

$$H^p_{n,m,\nu}(x) = \frac{(x^q-1)^{p(n-p)}}{\nu^p} \frac{d^p}{dx^p}[H_{n,m,\nu}(x)] \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

$$= n!\,(x^q-1)^{p(n-p)} \sum_{k=0}^{\left[\frac{n-p}{m}\right]} \frac{(-1)^k(\nu x)^{n-p-mk}}{k!\,(n-p-mk)!}.$$

Also

$$H^p_{n+p,m,\nu}(x) = (n+p)!\,(x^q-1)^{pn} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k(\nu x)^{n-mk}}{k!\,(n-mk)!} \quad \ldots \quad (2.2)$$

$$= \frac{(n+p)!}{n!}(x^q-1)^{pn} H_{n,m,\nu}(x) \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

where p and n are non-negative integers and m is a positive integer. If $p = 0$ and $m = \nu = 2$, $H^p_{n+p,m,\nu}(x)$ reduces to the Hermite polynomial $H_n(x)$.

Again (2.3) can also be written in the form of terminating series as follows:

$$H^p_{n+p,m,\nu}(x) = (n+1)_p(x^q-1)^{pn}(\nu x)^n\,{}_mF_0\left[\Delta(m\,;\,-n)\,;\,-\,;\,-\left(\frac{m}{\nu x}\right)^m\right] \quad (2.4)$$

and

$$H^p_{mm_1+m_2+p,m,\nu}(x) = \frac{(-1)^{m_1}}{m_1!\,(1)_{m_2}}(mm_1+m_2+p)!\,(\nu x)^{m_2}$$

$$\times (x^q-1)^{p(mm_1+m_2)}\,{}_2F_m\left[-m_{1,1}\,;\,\Delta(m\,;\,1+m_2)\,;\,\left(\frac{\nu x}{m}\right)^m\right]. \quad \ldots \quad (2.5)$$

3. Generating Relations

(i) From (2.2), we get

$$\sum_{n=0}^{\infty} \frac{H^{p}_{n+p,\,m,\,\nu}(x)t^n}{(n+p)\,!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k(x^q-1)^{pn}(\nu x)^{n-mk}t^n}{k\,!\;(n-mk)\,!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k(x^q-1)^{p(n+mk)}(\nu x)^n t^{n+mk}}{k\,!\;n\,!}$$

$$= e^{\{(x^q-1)^p \nu x t-(x^q-1)^{mp}t^m\}} \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

(ii) From (2.2), we have

$$\sum_{n=0}^{\infty} \frac{(C)_n H^{p}_{n+p,\,m,\,\nu}(x)t^n}{(n+p)\,!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^k(C)_n(x^q-1)^{pn}(\nu x)^{n-mk}t^n}{k\,!\;(n-mk)\,!}$$

$$= \sum_{n,\,k=0}^{\infty} \frac{(-1)^k(C)_{n+mk}(x^q-1)^{p(n+mk)}(\nu x)^n t^{n+mk}}{k\,!\;n\,!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(C)_{mk}(x^q-1)^{mpk}t^{mk}}{k\,!\;[1-\nu xt(x^q-1)^p]^{c+mk}} = [1-\nu xt(x^q-1)^p]^{-c}$$

$$\times {}_mF_0\left[\Delta(m\,;\,c)\,;\,-\,;\,-\left\{\frac{mt(x^q-1)^p}{1-\nu xt(x^q-1)^p}\right\}^m\right].$$

We thus arrive at the divergent generating relation

$$\{1-\nu xt(x^q-1)^p\}^{-c}\,{}_mF_0\left[\Delta(m\,;\,c)\,;\,-\,;\,-\left\{\frac{mt(x^q-1)^p}{1-\nu xt(x^q-1)^p}\right\}^m\right]$$

$$\cong \sum_{n=0}^{\infty} \frac{(C)_n H^{p}_{n+p,\,m,\,\nu}(x)t^n}{(n+p)\,!}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

4. nth Differential Formulae

By the use of (2.3), (1.3) and (1.4) respectively, we get nth differential formulae for $H^{p}_{n+p,\,m,\,\nu}(x)$ in the following two forms:

(i) $H^{p}_{n+p,\,m,\,\nu}(x) = (-1)^n(n+1)_p(x^q-1)^{pn}\nu^{(1-m)n}$

$$\times D^{(m-1)n}\,{}_1F_{m-1}\left[-n\,;\;\Delta[m(1)\,;\,1]\,;\;\left(\frac{\nu x}{m}\right)^m\right] \quad \ldots \quad \ldots \quad (4.1)$$

(ii) $$H^{p}_{n+p,\,m,\,\nu}(x) = \frac{\nu^n(p)_{n+1}(x^q-1)^{pn}}{m^{mn}\prod_{r=1}^{m-1}\left(\frac{r}{m}\right)_n}$$

$$\times D^{(m-1)n}\left[x^{mn}\,{}_mF_0\left[-n\,;\;\Delta\left[m(1)\,;\,1-\frac{1}{m}-n\right];\;-\,;\;-\left(-\frac{m}{\nu x}\right)^m\right]\right]$$

$$\ldots \quad (4.2)$$

5. Differential Equation

From (2.3) and (1.5), we shall derive the differential equation satisfied by $H^{p}_{n+p,m,\nu}(x)$ in the following manner.

Let us take

$$H_{n,m,\nu}(x) = Y(x^q-1)^{-pn}.$$

Therefore, (1.5) may be written as

$$m\sum_{r=0}^{m}[{}^mC_r D^{m-r}(x^q-1)^{-pn}D^rY]-\nu^m x(x^q-1)^{-pn}DY$$
$$+n\nu^m[pqx^q+(x^q-1)](x^q-1)^{-pn-1}Y = 0,$$

where

$$Y = H^{p}_{n+p,m,\nu}(x).$$

References

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