ABSOLUTE TAUBERIAN CONSTANTS FOR ABEL MEANS OF A FUNCTION

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This paper is concerned with introducing two estimates of the forms $A \leqslant BK(a)$ $A \leqslant CF(a)$, $(a > 0)$, where $A = \int_0^\infty |d\{f(x) - \psi(a/x)\}|$, $\psi(s) = s\int_0^\infty e^{-st} f(t)\,dt$ denote the Abel transform of the function $f(x) = \int_0^x g(t)\,dt$, $g(t)$ is of bounded variation in every finite interval of $t \geqslant 0$, $K(a)$, $F(a)$ are absolute Tauberian constants, $B = \int_0^\infty |d\{tg(t)\}| < \infty$, $C = \int_0^\infty |d\{\phi(t)\}| < \infty$ and $\phi(t) = 1/t\int_0^t ug(u)\,du$. The constants $K(a)$, $F(a)$ will be determined.

§1. We shall be concerned in this paper with absolute summability by the Abel method of a function. The definition of this method is as follows:

Definition—If

$$\psi(s) = s\int_0^\infty e^{-st} f(t)\,dt, \qquad \ldots(1.1)$$

and if $\psi(s)$ is of bounded variation in $(0, \infty)$, then the function $f(x)$ is said to be absolutely summable (A), or summable $|A|$.

This is the analogous definition of absolute summability (A) of a series $\sum_{n=0}^\infty a_n$, with partial sums $\{s_n\}$, which is given by Whittaker (1931).

In §§ 2, 3 of this paper, we shall introduce estimates of the forms

$$\int_0^\infty \left| d\left\{ f(x) - \psi\left(\frac{a}{x}\right)\right\}\right| \leqslant k(a) \int_0^\infty |d\{tg(t)\}|, \qquad (a > 0), \qquad \ldots(1.2)$$

and

$$\int_0^\infty \left| d\left\{ f(x) - \psi\left(\frac{a}{x}\right)\right\}\right| \leqslant F(a) \int_0^\infty |d\phi(t)|, \qquad (a > 0), \qquad \ldots(1.3)$$

where $g(t)$ is of bounded variation in every finite interval of $t \geqslant 0$,

$$f(x) = \int_0^x g(t)\, dt, \qquad \dots(1.4)$$

$$\phi(t) = \frac{1}{t} \int_0^t u g(u)\, du, \qquad \dots(1.5)$$

$K(c)$, $F(\alpha)$ are absolute Tauberian constants. The Tauberian conditions to be considered are

$$\int_0^\infty |\, d\{t g(t)\}\,| < \infty \qquad \dots(1.6)$$

or

$$\int_0^\infty |\, d\{\phi(t)\}\,| < \infty, \qquad \dots(1.7)$$

respectively.

Estimates of a similar type have been shown in Sherif (1972, 1974, 1976) for the absolute Cesàro means of a sequence; the absolute Hausdorff transformations and the absolute Cesàro means of a function respectively.

Again the estimates are analogous to those of various authors which appear in the bibliography of Sherif (1972) but for different summability methods instead; see for example Agnew (1952, 1954), Delange (1950) and Sherif (1963, 1965, 1967) respectively.

§2. *Theorem* 2.1—Suppose that (1.1), (1.4) and (1.6) hold. Then (1.2) holds, where for $\alpha > 0$,

$$K(\alpha) = \int_\alpha^\infty e^{-t} \frac{dt}{t} + \int_0^\alpha (1 - e^{-t}) \frac{dt}{t}. \qquad \dots(2.1)$$

This result is the best possible result in the sense that (1.2) becomes false if $K(\alpha)$ is replaced by any smaller constant.

The least possible value of $K(\alpha)$ occurs when

$$\alpha = \log 2. \qquad \dots(2.2)$$

For the proof of Theorem 2.1, we require the following lemma.

Lemma 2.1—Suppose that

$$G(x) = \int_0^\infty (x, t)\, dh(t),$$

where for every fixed x, $\chi(x, t)$ is continuous and bounded for $t \geqslant 0$, and $h(t)$ is of bounded variation in $(0, \infty)$. Then

$$\int_0^\infty |dG(x)| \leqslant A \int_0^\infty |dh(t)|, \qquad \ldots(2.3)$$

where

$$A = \sup_{0 \leqslant t} \int_0^\infty |d_x(x, t)|, \qquad \ldots(2.4)$$

and this constant A is the best possible in the sense that (2.3) becomes false if A is replaced by any smaller constant.

PROOF: Since $h(t)$ is of bounded variation, the assumption that $\chi(x, t)$ is continuous and bounded for fixed x ensures that

$$\int_0^\infty (x, t)\, dh(t)$$

converges, so that $G(x)$ is defined. If $0 = x < x_1 < x_2 < \ldots$ then

$$\sum_{\nu=1}^{n} |G(x_\nu) - G(x_{\nu-1})|$$

$$\leqslant \int_0^\infty \{\sum_{\nu=1}^{n} |\chi(x_\nu, t) - (x_{\nu-1}, t)|\} |dh(t)|$$

$$\leqslant \int_0^\infty |dh(t)| \int_0^\infty |d_x\chi(x, t)|$$

$$\leqslant A \int_0^\infty |dh(t)|.$$

Hence $G(x)$ is of bounded variation in $(0, \infty)$, and its variation does not exceed $A \int_0^\infty |dh(t)|$, in other words (2.3) holds.

Conversely, given $\epsilon > 0$ there is a t_0 such that

$$\int_0^\infty |d_x\chi(x, t)| > A - \epsilon,$$

choose

$$h(t) = \begin{cases} 0 & (t \leqslant t_0), \\ 1 & (t > t_0); \end{cases}$$

then

$$\int_0^\infty |dh(t)| = 1;$$

but

$$G(x) = \chi(x, t_0),$$

so that

$$\int_0^\infty |dG(x)| > A - \epsilon.$$

The conclusion thus follows.

PROOF OF THEOREM 2.1: Since

$$g(u) = \frac{1}{u} \int_0^u d(tg(t)), \qquad \ldots(2.5)$$

it thus follows from (1.4) that

$$f(x) = \int_0^x \frac{du}{u} \int_0^u d(tg(t))$$

$$= \int_0^x d(tg(t)) \int_t^x \frac{du}{u}$$

$$= \int_0^x \left(\log\left(\frac{x}{t}\right) d(tg(t))\right). \qquad \ldots(2.6)$$

Also, it follows from (1.1) that

$$\psi\left(\frac{a}{x}\right) = \frac{a}{x} \int_0^\infty e^{-au/x} f(u)\, du. \qquad \ldots(2.7)$$

Integrating (2.7) by parts we have

$$\psi\left(\frac{a}{x}\right) = \int_0^\infty e^{-au/x} g(u)\, du. \qquad \ldots(2.8)$$

Substituting with (2.5) in (2.8), we have

$$\psi\left(\frac{a}{x}\right) = \int_0^\infty e^{-au/x} \frac{du}{u} \int_0^u d(tg(t))$$

$$= \int_0^\infty d(tg(t)) \int_t^\infty e^{-au/x} du. \qquad \ldots(2.9)$$

It is clear from (2.6) that

$$f(x) = \int_0^x d\,(tg\,(t)) \log\left(\frac{x}{t}\right). \qquad \ldots(2.10)$$

Then, it follows from (2.9) and (2.10) that

$$f(x) - \psi\left(\frac{a}{x}\right)$$

$$= \int_0^x d\,(tg\,(t)) \log\left(\frac{x}{t}\right) - \int_0^\infty d\,(tg\,(t)) \int_t^\infty e^{-au/x} \frac{du}{u}. \qquad \ldots(2.11)$$

Now (2.11) is a transformation of the type considered in Lemma 2.1. Thus, for $a > 0$, the $\chi(x, t)$ of Lemma 2.1 is equal to

$$\left.\begin{array}{ll} \log\left(\frac{x}{t}\right) - \int_t^\infty e^{-au/x} \frac{du}{u}, & \text{for } (0 < t < x), \\ - \int_t^\infty e^{-au/x} \frac{du}{u}, & \text{for } (t \geqslant x). \end{array}\right\} \qquad \ldots(2.12)$$

Now

$$\frac{\partial}{\partial x}\left\{ \int_t^\infty e^{-au/x} \frac{du}{u}\right\} = \frac{a}{x^2} \int_t^\infty e^{-au/x}\, du$$

$$= \frac{1}{x} e^{-at/x}. \qquad \ldots(2.13)$$

Thus, it follows from (2.12) and (2.13) that for fixed t, $\chi(x, t)$ is an indefinite integral with respect to x of $\chi_*(x, t)$ where

$$\chi_*(x, t) = \begin{cases} \frac{1}{x}(1 - e^{-at/x}), & \text{for } (0 < t < x), \\ -\frac{1}{x} e^{-at/x}, & \text{for } (t \geqslant x). \end{cases} \qquad \ldots(2.14)$$

Applying Lemma 2.1, it follows from (2.14) that

$$K(a) = \sup_{0 \leqslant t}\left[\int_0^t e^{-at/x} \frac{dx}{x} + \int_t^\infty (1 - e^{-at/x}) \frac{dx}{x}\right]. \qquad \ldots(2.15)$$

Putting $at/x = u$ in (2.15), it is easily seen that the result (1.2) holds with $K(\alpha)$ given by (2.1), together with the assertion that this is the best possible result follow at once from Lemma 2.1. We now prove equation (2.2). It is plain from (2.1) that

$$\frac{dk(\alpha)}{d\alpha} = \frac{1}{\alpha}(1 - 2e^{-\alpha}). \qquad ...(2.16)$$

It is then clear that the expression in brackets of (2.16) is negative when $e^{-\alpha} > \frac{1}{2}$ and is positive when $e^{-\alpha} < \frac{1}{2}$. It vanishes when $e^{-\alpha} = \frac{1}{2}$ from which (2.2) holds.

§3. *Theorem* 3.1—Suppose that (1.1), (1.4), (1.5) and (1.7) hold. Then (1.3) holds, where for $\alpha > 0$,

$$F(\alpha) = 1 + \int_{\alpha}^{\infty} e^{-t}(t+1)\frac{dt}{t} + \int_{0}^{\infty} \left(1 - (t+1)e^{-t}\right)\frac{dt}{t}. \qquad ...(3.1)$$

This result is the best possible result in the sense that (1.3) becomes false if $F(\alpha)$ is replaced by any smaller constant.

The least possible value of $F(\alpha)$ occurs when $\alpha = \alpha_1$, where α_1 is the unique positive solution of the equation

$$1 - 2(1 + \alpha)e^{-\alpha} = 0. \qquad ...(3.2)$$

PROOF: Using (1.5), we see that $\phi(t)$ is an indefinite integral of the function $\phi'(t)$ defined by

$$\phi'(t) = g(t) - \frac{1}{t^2}\int_{0}^{t} ug(u)\,du.$$

Thus,

$$g(t) = \phi'(t) + \frac{\phi(t)}{t}$$

$$= \phi'(t) + \frac{1}{t}\int_{0}^{t}\phi'(x)\,dx. \qquad ...(3.3)$$

Substituting with (3.3) in (1.4) we get

$$f(x) = \int_{0}^{x}\phi'(t)\,dt + \int_{0}^{x}\frac{d\omega}{\omega}\int_{0}^{\omega}\phi'(t)\,dt. \qquad ...(3.4)$$

Inverting the order of integration in the second integral of (3.4), and then evaluating the inner integral, we get

$$f(x) = \int_{0}^{x}\left(1 + \log\left(\frac{x}{t}\right)\right)\phi'(t)\,dt. \qquad ...(3.5)$$

Also, it follows from (3.3) and (2.8) that

$$\psi\left(\frac{a}{x}\right) = \int_0^\infty e^{-at/x} \left\{\phi'(t) + \frac{1}{t}\int_0^t \phi'(x)\,dx\right\} dt$$

$$= \int_0^\infty e^{-at/x}\phi'(t)\,dt + \int_0^\infty e^{-au/x}\frac{du}{u}\int_0^u \phi'(t)\,dt. \qquad \ldots(3.6)$$

Inverting the order of integration in the second integral of (3.6) we get

$$\psi\left(\frac{a}{x}\right) = \int_0^\infty e^{-at/x}\phi'(t)\,dt + \int_0^\infty \phi'(t)\,dt\int_t^\infty e^{-au/x}\frac{du}{u}. \qquad \ldots(3.7)$$

Thus, it follows from (3.5) and (3.7) that

$$f(x) - \psi\left(\frac{a}{x}\right)$$

$$= \int_0^x \left(1 + \log\left(\frac{x}{t}\right)\right)\phi'(t)\,dt$$

$$- \int_0^\infty \left\{e^{-at/x} + \int_t^\infty e^{-au/x}\frac{du}{u}\right\}\phi'(t)\,dt. \qquad \ldots(3.8)$$

Now, (3.8) is a transformation of the type considered in Lemma 2.1. Thus, the $\chi(x, t)$ of Lemma 2.1 is equal to

$$\begin{cases} 1 + \log\left(\frac{x}{t}\right) - e^{-at/x} - \int_t^\infty e^{-au/x}\frac{du}{u}, & \text{for } (0 < t < x), \\ e^{-at/x} - \int_t^\infty e^{-au/x}\frac{du}{u}, & \text{for } (t > x). \end{cases}$$

Now,

$\frac{\partial}{\partial x}\chi(x, t)$ is equal to

$$\left.\begin{array}{l} \frac{1}{x}\left(1 - \frac{at}{x}e^{-at/x} - e^{-at/x}\right), \text{ for } (0 < t < x), \\ -\frac{e^{-at/x}}{x}\left(\frac{at}{x} + 1\right), \text{ for } (t > x). \end{array}\right\} \qquad \ldots(3.9)$$

We further note that $\chi(x, t)$ has a jump of 1 at $x = t$. Thus, it follows from Lemma 2.1 that

$$F(\alpha) = \sup_{0 \leqslant t} \left[\int_0^t e^{-\alpha t/x} \left(\frac{\alpha t}{x} + 1 \right) \frac{dx}{x} \right.$$

$$\left. + \int_t^\infty \left| \frac{1}{x} \left(1 - \frac{\alpha t}{x} e^{-\alpha t/x} - e^{\alpha t/x} \right) \right| dx \right]. \qquad \ldots(3.10)$$

Now the expression inside the modulus sign in (3.10) is equal to

$$\frac{e^{-\alpha t/x}}{x^2} (x\, e^{\alpha t/x} - \alpha t - x). \qquad \ldots(3.11)$$

Putting $x = \alpha t/y$, (3.11) becomes

$$\frac{y}{\alpha t} e^{-y} (e^y - y - 1). \qquad \ldots(3.12)$$

It is thus clear that (3.12) is positive for $y > 0$. Then, we can omit the modulus sign in (3.10). On doing this, we have

$$\left\{ \begin{aligned} F(\alpha) = \sup_{0 \leqslant t} & \left[\int_0^t e^{-\alpha t/x} \left(\frac{\alpha t}{x} + 1 \right) \frac{dx}{x} \right. \\ & \left. + \int_t^\infty \left(1 - \frac{\alpha t}{x} e^{-\alpha t/x} - e^{-\alpha t/x} \right) \frac{\alpha x}{x} + 1 \right]. \end{aligned} \right. \qquad \ldots(3.13)$$

Putting $\alpha t/x = u$ in (3.13), the result that (1.3) holds with $F(\alpha)$ given by (3.1) together with the assertion that this is the best possible result follow at once from Lemma 2.1. We now prove (3.2).

$$\frac{dF(\alpha)}{d\alpha} = -\frac{e^{-\alpha}}{\alpha} (\alpha + 1) + \left(1 - (\alpha + 1) e^{-\alpha}\right) \frac{1}{\alpha}$$

$$= \frac{1}{\alpha} \{1 - 2(\alpha + 1) e^{-\alpha}\}. \qquad \ldots(3.14)$$

But, we can verify by differentiation that the expression in curly brackets in (3.14) is increasing for $\alpha > 0$. It tends to -1 as $\alpha \to 0$ and to 1 as $\alpha \to \infty$ and has thus a unique zero (say $\alpha = \alpha'$) for $\alpha > 0$, and the minimum of $F(\alpha)$ occurs when $\alpha = \alpha'$ is the unique solution of equation (2.2).

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