HEAT CONDUCTION IN A TRUNCATED WEDGE OF FINITE HEIGHT

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The problem of finding the temperature at any point of a truncated wedge of finite height when there are sources of heat within it, for boundary conditions of the radiation type is considered. The different types of finite integral transforms are employed to solve the problem.

1. INTRODUCTION

Marchi and Zgrablich (1964) have solved the problem of finding the temperature at any point of a hollow cylinder of any height with heat radiation on its surfaces by the extended finite Hankel transform and Sine transform. Mathur (1971) has obtained the temperature at any point in finite solid cylinder and hollow cylinder when there are sources of heat within it, for the boundary conditions of radiation type. Mehta (1969) tackled some time reversal heat conduction problems with the help of integral transforms for cylindrical shell of infinite height with heat generation and radiation, truncated wedge of finite height and semi-infinite solid containing an exterior plane crack with a circular boundary and infinitely long cylindrical cavity.

The object of this paper is to solve a more general problem which shall enable us to find the temperature at any point of a truncated wedge of finite height with heat radiation on its outside and inside surfaces and having sources of heat within it. The problem considered here is more general in nature than that considered by Marchi and Zgrablich (1964) and Mathur (1971). We suppose that the media bounding the two surfaces are in general different and all functions involved satisfy Dirichlet's conditions in the intervals considered. Later on, we obtain the temperature at any point on the boundary considered with the help of special functions.

2. FORMULATION OF THE PROBLEM

We consider the distribution of heat in a truncated wedge of finite height defined as $0 \leqslant z \leqslant h$, $a \leqslant r \leqslant b$, $0 \leqslant \theta \leqslant \xi$ having sources of heat within it. There is heat radiation on its surfaces $r = a$ and $r = b$. The temperature $u \equiv u(r, z, \theta, t)$ at any point of the wedge, where t is the time will be the solution of the conduction equation [Mehta 1969, p. 400, (4.1)]:

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right) + \phi\,(r, \theta, z, t), \quad \text{...(2.1)}$$

$$0 < z < h,\ a < r < b,\ 0 < \theta < \xi$$

where a is a constant known as diffusivity. The boundary conditions of the problem are

$$k_1 \frac{\partial u}{\partial r} + u = F(\theta, z, t);\ \ r = a, 0 < z < h, 0 < \theta < \xi, t > 0 \quad \text{...(2.2)}$$

$$k_2 \frac{\partial u}{\partial r} + u = H(\theta, z, t);\ \ r = b, 0 < z < h,\ 0 < \theta < \xi,\ t > 0 \quad \text{...(2.3)}$$

where k_1 and k_2 are radiation constants.

$$u = f(r, t)\ ;\ \ z = h, 0 < \theta < \xi,\ a < r < b, t > 0 \quad \text{...(2.4)}$$

$$u = g(r, t)\ ;\ \ z = 0,\ 0 < \theta < \xi,\ a < r < b,\ t > 0 \quad \text{...(2.5)}$$

$$u \to 0\ ;\ \ \theta = 0,\ 0 < z < h,\ a < r < b,\ t > 0 \quad \text{...(2.6)}$$

$$u \to 0\ ;\ \ \theta = \xi,\ 0 < z < h,\ a < r < b,\ t > 0 \quad \text{...(2.7)}$$

$$u = G(r, \theta, z);\ \ t = 0,\ 0 < z < h, a < r < b,\ 0 < \theta < \xi. \quad \text{...(2.8)}$$

3. Results Required in Solving the Problem (2.1)

(*a*) Marchi and Zgrablich [1964, p. 160, (8)] have defined the following integral transform of a function $f(x)$:

$$\bar{f}(n) = \int_a^b x f(x)\, S_p(\mu_n x)\, dx \quad \text{...(3.1)}$$

where $S_p(\mu_n x) \equiv S_p(k_1, k_2, \mu_n x)$ and μ_n is chosen as a positive root of the equation

$$J_p(k_1, \mu a)\ G_p(k_2, \mu b) - J_p(k_2, \mu b)\ \ G_p(k_1, \mu a) = 0 \quad \text{...(3.2)}$$

and the inversion transform is given by

$$f(x) = \sum_n \frac{1}{C_n}\, \bar{f}(n)\, S_p(\mu_n x) \quad \text{...(3.3)}$$

the summation being taken over the positive root of (3.2) and the value of C_n is same as given by Marchi and Zgrablich [1964, p. 160, (11)].

(*b*) The basic property of the transform (3.1) to be used is [Marchi and Zgrablich 1964, p. 161, (14)]

$$\int_a^b x\left(\frac{\partial^2 f}{\partial x^2}+\frac{1}{x}\frac{\partial f}{\partial x}-\frac{p^2}{x^2}f\right) S_p(\mu_n x)\,dx$$

$$=\frac{b}{k_2}S_p(\mu_n b)\left[f+k_2\frac{\partial f}{\partial x}\right]_{x=b}-\frac{a}{k_1}S_p(\mu_n a)\left[f+k_1\frac{\partial f}{\partial x}\right]_{x=a}$$

$$-\mu_n^2\bar{f}(n). \qquad \ldots(3.4)$$

(*c*) The finite sine transform of a function $f(x)$ has been defined as (Sneddon 1951, p. 74):

$$f_s(m)=\int_0^h f(x)\sin\frac{m\pi x}{h}\,dx \qquad \ldots(3.5)$$

where $f(x)$ is given by $f(x)=\frac{2}{h}\sum_{m=1}^{\infty} f_s(m)\sin\frac{m\pi x}{h}$. $\qquad \ldots(3.6)$

(*d*) The following property of finite sine transform (Sneddon 1951, p. 75) will also be required in the sequel

$$\int_0^h \frac{\partial^2 f}{\partial x^2}\sin\frac{m\pi x}{h}\,dx=\frac{m\pi}{h}\left[(-1)^{m+1}f_s(m,h,t)+f_s(m\ 0,t)\right]$$

$$-\frac{m^2\pi^2}{h^2}f_s(m). \qquad \ldots(3.7)$$

4. Solution of the Problem

Multiplying both the sides of (2.1) by $\sin m_1\theta\pi/\xi$, integrating the result thus obtained with respect to θ between the limits $(0,\xi)$, we get, in view of (3.7), (2.6) and (2.7), the following differential equation:

$$\frac{\partial u_s}{\partial t}=\alpha\left(\frac{\partial^2 u_s}{\partial r^2}+\frac{1}{r}\frac{\partial u_s}{\partial r}-\frac{p^2}{r^2}u_s+\frac{\partial^2 u}{\partial z^2}\right)+\phi_s(r,m_1,z,t) \qquad \ldots(4.1)$$

where

$$u_s=u_s(r,m_1,z,t)=\int_0^\xi u(r,z,\theta,t)\sin\frac{m_1\pi\theta}{\xi}\,d\theta \text{ and } p=\frac{m_1\pi}{\xi}.$$

Now taking transform defined by (3.1) on both the sides of (4.1) and using (3.4) in it, we get, in view of boundary conditions (2.2) and (2.3) the following result:

$$\frac{\partial \bar{u}_s}{\partial t} = aX(z, t) - a\mu_n^2 \bar{u}_s + a \frac{\partial^2 \bar{u}_s}{\partial z^2} + \bar{\phi}_s(n, m_1, z, t) \qquad \ldots(4.2)$$

where

$$\bar{u}_s \equiv \bar{u}_s(n, m_1, z, t) = \int_a^b r u_s(r, m_1, z, t)\, S_p(\mu_n r)\, dr$$

$$X(z, t) = \frac{b}{k_2} S_p(\mu_n b)\, H_s(m_1, z, t) - \frac{a}{k_1} S_p(\mu_n a)\, F_s(m_1, z, t) \qquad \ldots(4.3)$$

$H_s(m_1, z, t)$ and $F_s(m_1, z, t)$ are the sine transforms of $H(\theta, z, t)$ and $F(\theta, z, t)$ respectively with respect to θ.

Again multiplying both the sides of (4.2) by $\sin m_2\pi z/h$, integrating with respect to z between the limits $(0, h)$ and using (2.4) and (2.5), we get

$$\frac{d\bar{u}_{s,s}}{dt} = a\, X_s(m_2, t) - a\mu_n^2\, \bar{u}_{s,s} - a\beta\; Y(t) - a\, \frac{m_2^2\pi^2}{h^2}\, \bar{u}_{s,s}$$

$$+ \bar{\phi}_{s,s}(n, m_1, m_2, t) \qquad \ldots(4.4)$$

$$\bar{u}_{s,s} \equiv \bar{u}_{s,s}(n, m_1, m_2, t) = \int_0^h \bar{u}_s(n, m_1, z, t) \sin \frac{m_2\pi z}{h}\, dz,$$

$$X_s(m_2, t) = \int_0^h X(z, t)\, \sin \frac{m_2\pi z}{h}\, dz; \quad Y(t) = (-1)^{m_2} \bar{f}(n, t) - \bar{g}(n, t)$$

[$\bar{f}(n, t)$ and $\bar{g}(n, t)$ are extended finite Hankel transform given by (3.1) of the functions $f(r, t)$ and $g(r, t)$ respectively] and

$$\beta = \frac{m_2\xi}{m_1 h}\, \{1 - (-1)^{m_1}\}.$$

Now making use of Laplace transform and taking in view the initial condition (2.8), eqn. (4.4) is reduced to

$$(q + A)\, L\{\bar{u}_{s,s}\} = \bar{G}_{s,s}(n, m_1, m_2) + aL\{X_s(m_2, t)\} - a\beta\; L\{Y(t)\}$$

$$+ L\{\bar{\phi}_{s,s}(n, m_1, m_2, t)\} \qquad \ldots(4.5)$$

where

$$L\{\bar{u}_{s,s}\} = \int_0^\infty e^{-qt}\, \bar{u}_{s,s}\, dt \text{ and } A = a\left(\mu_n^2 + \frac{m_2^2\pi^2}{h^2}\right)$$

and

$$\bar{G}_{s,s}(n, m_1, m_2) = \int_0^h \int_0^b \int_0^\xi r\, G(r, \theta, z) \sin \frac{m_1 \pi \theta}{\xi}$$

$$\times\, S_p(\mu_n r) \sin \frac{m_2 \pi z}{h}\, d\theta\, dr\, dz.$$

Further taking the inverse Laplace transform of (4.5) and making use of convolution theorem we obtain

$$\bar{u}_{s,s}(n, m_1, m_2, t) = \bar{G}_{s,s}(n, m_1, m_2)\, e^{-At} + a \int_0^t X_s(m_2, T)\, e^{-A(t-T)}\, dT$$

$$- a\beta \int_0^t Y(T)\, e^{-A(t-T)}\, dT + \int_0^t \bar{\phi}_{s,s}(n, m_1, m_2, T)\, e^{-A(t-T)}\, dT. \qquad \ldots(4.6)$$

Lastly by the application of the inversion formulae (3.3) and (3.6), we get the solution of the boundary value problem:

$$u(r, \theta, z, t) = \frac{4}{h\xi} \sum_{m_1=1}^{\infty} \sum_{n} \sum_{m_2=1}^{\infty} \frac{1}{C_n} \sin \frac{m_2 \pi z}{h} \sin \frac{m_1 \pi \theta}{\xi}\, S_p(\mu_n r)$$

$$\times\, \bar{u}_{s,s}(n, m_1, m_2, t). \qquad \ldots(4.7)$$

Corollary—Taking u and ϕ as independent of θ in (2.1) and adjusting the boundary conditions accordingly, we have the conduction equation

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) + \phi(r, z, t), \qquad \ldots(4.8)$$

$$a < r < b,\ 0 < z < h$$

with the boundary conditions

$$k_1 \frac{\partial u}{\partial r} + u = F(z, t),\ r = a,\ 0 < z < h,\ t > 0;$$

$$k_2 \frac{\partial u}{\partial r} + u = H(z, t),\ r = b,\ 0 < z < h,\ t > 0;$$

$$u = f(r, t),\ z = h,\ a < r < b,\ t > 0;$$

$$u = g(r, t),\ z = 0,\ a < r < b,\ t > 0;$$

$$u = G(r, z),\ t = 0,\ a < r < b,\ 0 < z < h.$$

On using the extended finite Hankel transform (3.1) with $p = 0$ and the finite sine transform (3.5) with respect to z between $(0, h)$ in (4.8) and following the same method as given above, the temperature at any point of finite hollow cylinder defined by $0 < z < h$, $a < r < b$ having symmetry about the axis of the cylinder and sources of heat within it with heat radiation on its outside and inside surfaces is given by

$$u(r, z, t) = \frac{2}{h} \sum_{m_2=1}^{\infty} \sum_{n} \frac{1}{C_n} \sin \frac{m_2 \pi z}{h} S_0(\mu_n r) \bar{u}_s(n, m_2, t) \qquad \ldots(4.9)$$

where

$$\bar{u}_s(n, m_1, m_2, t) = \bar{G}_s(n, m_2) e^{-At} + a \int_0^t X_s'(m_2, t) e^{-A(t-T)} dT$$

$$- \frac{a m_2 \pi}{h} \int_0^t Y'(T) e^{-A(t-T)} dT + \int_0^t \bar{\phi}_s(n, m_2, T) e^{-A(t-T)} dT$$

and $X_s'(m_2, T)$; $Y'(T)$ are the values of $X_s(m_2, T)$; $Y(T)$ [given with (4.4)] with $p = 0$.

If $g(r, t)$ is taken as constant (say u_0), the above problem reduces to that of [Mathur 1971, p. 24, (2.1)] which in itself includes the problem considered by Marchi and Zgrablich (1964). Hence the problem considered here can be claimed to be more general and includes all possible problems on boundary conditions having circular symmetry.

5. Examples

Example (i)—Taking $F(\theta, z, t) = f(r, t) = g(r, t) = \phi(r, \theta, z, t) = 0$;

$$H(\theta, z, t) = D(1 - e^{-Bt});$$

$$G(r, \theta, z) = r^{Q-1} (\xi^2 - \theta^2)^{M-\frac{1}{2}} (hz - z^2)^{P-\frac{1}{2}} [\mathrm{Re}(M) > -\tfrac{1}{2},$$

$$\mathrm{Re}(P) > -\tfrac{1}{2}].$$

in (2.2) to (2.8) and evaluating the integrals involved in (4.6) with the help of known results [Erdelyi *et al.* 1954 *a*, p. 69, (7) and (13); Erdelyi *et al.* 1953, p. 90, (7)], we get the temperature at any point in the truncated wedge,

$$u(r, \theta, z, t) = 4 \sum_{m_1=1}^{\infty} \sum_{n} \sum_{m_2=1}^{\infty} \frac{1}{C_n} \sin \frac{m_1 \pi \theta}{\xi} \sin \frac{m_2 \pi z}{h}$$

$$\times S_p(\mu_n r) \bar{u}_{s,c}(n, m_1, m_2, t) \qquad \ldots(5.1)$$

where

$$\bar{u}_{s,s}(n, m_1, m_2, t)$$
$$= R_4 \frac{e^{-At}(\xi)^{2M-1}(h)^{2P-1}\Gamma(P+\frac{1}{2})\sin(\frac{1}{2}m_2\pi)J_v(\frac{1}{2}m_2\pi)}{(\pi)^{M+P-\frac{1}{2}}(m_1)^M(m_2)^P}$$
$$\times S_{M,M}(m_1\pi) + \frac{DKab\, S_p(\mu_n b)[A(1-e^{-Bt}) - B(1-e^{-At})]}{k_2 A(A-B)} \quad \ldots(5.2)$$

where $s_{M,M}(z)$ is the Lommel's function [Erdelyi *et al.* 1953, p. 372] and

$$R_4 = R_3(\mu_n)^{-Q}[b\{(p+Q-1)J_p(\mu_n b)S_{Q-1,\,p-1}(\mu_n b)$$
$$- J_{p-1}(\mu_n b)S_{Q,P}(\mu_n b)\} - a\{(p+Q-1)J_p(\mu_n a)S_{Q-1,\,p-1}(\mu_n a)$$
$$- J_{p-1}(\mu_n a)S_{Q,P}(\mu_n a)\}]$$
$$R_3 = R_1 - \tfrac{1}{2}R_2 \operatorname{cosec}(p\pi)\ \{(-1)^p - e^{ip\pi}\};$$
$$R_1 = G_p(k_1, \mu_n a) + G_p(k_2, \mu_n b)\ ;\ R_2 = J_p(k_1, \mu_n a) + J_p(k_2, \mu_n b);$$
$$J_p(k_i, \mu_n x) = J_p(\mu_n x) + k_i\mu_n J_p'(\mu_n x);$$
$$G_p(\mu_n x) = \tfrac{1}{2}\operatorname{cosec}(p\pi)(J_{-p}(\mu_n x) - e^{i\pi p}J_p(\mu_n x));$$

and

$$K = \frac{((-1)^{m_1}-1)((-1)^{m_2}-1)}{m_1 m_2 (\pi)^2}.$$

Example (ii)—If we take

$$H(\theta, z, t) = g(r, t) = \phi(r, \theta, z, t) = 0$$
$$F(\theta, z, t) = e^{-Bt}\theta\ (\xi^2 - \theta^2)^{M-\frac{1}{2}}(h^2 - z^2)^{P-\frac{1}{2}}, [\mathrm{Re}(M) > -\tfrac{1}{2},\ \mathrm{Re}(P) > -\tfrac{1}{2}]$$
$$f(r, t) = r^{Q-1}\delta(t);\ [\delta(t) \text{ is the Dirac delta function}]$$
$$G(r, \theta, z) = r^{Q-1}z^{v-\frac{1}{2}}(h^2 - z^2)^w, [\mathrm{Re}(w) > -1,\ \mathrm{Re}(v) > -3/2]$$

in eqns. (2.2) to (2.8) and then evaluate the corresponding integrals involved in (4.6) by making use of the results [Erdelyi *et al.* 1953, p. 90, (7); Erdelyi *et al.* 1954 *a*, p. 69, (7) & (9); Erdelyi *et al.* 1954, p. 26, (34)], we get

$$u(r, \theta, z, t) = \frac{4}{h\xi}\sum_{m_1=1}^{\infty}\sum_{n}\sum_{m_2=1}^{\infty}\frac{1}{C_n}\sin\frac{m_1\pi\theta}{\xi}\sin\frac{m_2\pi z}{h}\ s_p(\mu_n r)$$
$$\times e^{-At}\bar{u}_{s,s}(n, m_1, m_2, t) \quad \ldots(5.3)$$

where

$$\bar{u}_{s,s}(n, m_1, m_2, t)$$

$$= \frac{R_4}{2}(h)^{2w+v+3/2} B\left(w+1, \frac{2v+3}{4}\right)\frac{m_2}{m_1}\left(1-(-1)^{m_1}\right)$$

$$\times {}_1F_2\left(\frac{2v+3}{4}; \frac{3}{2}; \frac{7+2v+4w}{4}; -\frac{m_2^2\pi^2}{4}\right)$$

$$-\frac{\alpha a\, S_p(\mu_n a)\left[e^{(A-B)t}-1\right] 2^{M-1}(\xi)^{2M}}{k_1(A-B)(\pi)^{M+P+\frac{1}{2}}(m_1)^M (m_2)^P}$$

$$\times (h)^{2P}\, \Gamma\left(M+\tfrac{1}{2}\right) s_{P,P}(m_2\pi)\, J_{M+1}(m_1\pi) + (-1)^{m_2+1}\alpha\beta R_4 \qquad \ldots(5.4)$$

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