ON SOME RESULTS CONCERNING GENERALISED *H*-FUNCTION OF TWO VARIABLES

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An illuminating discussion on the convergence conditions of the generalised *H*-function of two variables has been incorporated to fill the gaps in the existing literature. In the sequel, a multiplication theorem and an inter-relation are also established.

1. INTRODUCTION

In sequel to Agarwal (1965) and Sharma's (1965) extension to two variables of Meijer's *G*-function, several authors, notably, Munot and Kalla (1971), Verma (1971), Agarwal and Mathur (1968) and Chaturvedi and Goyal (1971) have almost simultaneously followed the suit in generalising Fox's *H*-function to what we refer as the generalised *H*-function of two variables. The definitions employed by the authors are slightly at variance with each other although the function considered in each case is essentially the same. We shall give here a definition which, it will be seen, is compact. self-explanatory and remove the shortcomings of the earlier definitions so far considered. In our notation

$$H^{m_1, 0;\, m_2, n_2;\, m_3, n_3}_{p_1, q_1;\, p_2, q_2;\, p_3, q_3}\left[\begin{array}{c|c} x & \{(a_{p_1}, A_{p_1})\} : \{(b_{q_1}, B_{q_1})\} \\ & \{(c_{p_2}, C_{q_2})\} : \{(d_{q_2}, D_{q_2})\} \\ y & \{(e_{p_3}, E_{p_3})\} ; \{(f_{q_3}, F_{q_3})\} \end{array}\right]$$

$$= H\left[\begin{array}{c|c} x & (a, A); (b, B) \\ & (c, C); (d, D) \\ y & (e, E); (f, F) \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} F(\xi + \eta)\, \psi(\xi)\, \phi(\eta)\, d\xi\, d\eta. \qquad \ldots(1.1)$$

The sequence $\{(a_{p_1}, A_{p_1})\}$ will stand for the set of p_1-ordered pairs and it shall be abbreviated by (a, A) and so on. Furthermore, the indices and the parameters will be omitted whenever the context is clear so that one could denote the function simply by the notation $H\begin{bmatrix} x \\ y \end{bmatrix}$. The functions $F(\xi+\eta)$, $\psi(\xi)$ and $\phi(\eta)$ denote respectively the Γ products:

$$\left.\begin{aligned} F(\xi+\eta) &= \frac{\prod_{j=1}^{m_1} \Gamma(a_j + A_j\xi + A_j\eta)}{\prod_{j=m_1+1}^{p_1} \Gamma(1-a_j-A_j\xi-A_j\eta) \prod_{j=1}^{q_1} \Gamma(b_j+B_j\xi+B_j\eta)} \\ \psi(\xi) &= \frac{\prod_{j=1}^{m_2} \Gamma(1-c_j+C_j\xi) \prod_{j=1}^{n_2} \Gamma(d_j-D_j\xi)}{\prod_{j=1}^{p_2} \Gamma(c_j-C_j\xi) \prod_{j=n_2+1}^{q_2} \Gamma(1-d_j+D_j\xi)} x^{\xi} \\ \phi(\eta) &= \frac{\prod_{j=1}^{m_3} \Gamma(1-e_j+E_j\eta) \prod_{j=1}^{n_3} \Gamma(f_j-F_j\eta)}{\prod_{j=m_3+1}^{p_3} \Gamma(e_j-E_j\eta) \prod_{j=n_3+1}^{q_3} \Gamma(1-f_j+F_j\eta)} y^{\eta} \end{aligned}\right\} \quad \ldots(1.2)$$

where x and y are not equal to zero and the empty product is interpreted as unity. The non-negative integers m_1, m_2, m_3, n_2, n_3, p_1, p_2, p_3, q_1, q_2 and q_3 satisfy the inequalities:

$$p_1 \geqslant m_1 \geqslant 0, \quad p_2 \geqslant m_2 \geqslant 0, \quad p_3 \geqslant m_3 \geqslant 0, \quad q_1 \geqslant 0,$$

$$q_2 \geqslant n_2 \geqslant 0, \quad q_3 \geqslant n_3 \geqslant 0, \quad q_1+q_2 \geqslant p_1+p_2, \quad q_1+q_3 \geqslant p_1+p_3.$$

The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(d_j - D_j\xi)$, $j = 1, \ldots, n_2$, lie to the right and the poles of $\Gamma(1-c_j+C_j\xi)$, $j = 1, \ldots, m_2$ and $\Gamma(a_j + A_j\xi + A_j\eta)$, $j = 1, \ldots, m_1$, to the left of the contour.

Similarly, the contour L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$, $j = 1, \ldots, n_3$, lie to the right and the poles of $\Gamma(1-e_j+E_j\eta)$, $j = 1, \ldots, m_3$ and $\Gamma(a_j + A_j\xi + A_j\eta)$, $j = 1, \ldots, m_1$, to the left of the contour.

All the a's, b's, c's, d's, e's and f's are either real or complex numbers whereas all the A's, B's, C's, D's, E's and F's are real and positive. It is also to be understood that the sequence of parameters $\{(a_{m_1}, A_{m_1})\}$, $\{(1-c_{m_2}, C_{m_2})\}$, $\{(d_{n_2}, D_{n_2})\}$, $\{(1-e_{m_3}, E_{m_3})\}$ and $\{(f_{n_3}, F_{n_3})\}$ are all such that none of the poles of the integrand coincide.

2. Convergence of the Integrals

The integral which we have defined in (1.1) converges under certain conditions. These conditions can be easily obtained by considering the behaviour of the integrand over the closed contours. We shall follow the method suggested by MacRobert (1958):

Section I

We first consider the integral in the plane taken round the closed contour L_1, consisting of the imaginary axis from $-iR_1$ to $+iR_1$, where R_1 is large and that part of the semi-circle $|\xi| = R_1$ which lies to the right of the imaginary axis, R_1 being so chosen that the circle always passes between the poles of the integrand.

In the investigation, we require the gamma-function formula

$$\Gamma(z)\,\Gamma(1-z) = \pi/\sin(\pi z), \qquad \ldots(2.1)$$

and the formula, MacRobert (1958), p. 374, *i.e.*, if

$$|\arg z| \leqslant \pi - \delta,\ \delta \text{ being a positive number such that } 0 < \delta < \pi,$$

$$|\Gamma(z+\nu)| \leqslant M\,|z|^{g-\frac{1}{2}} \exp[x \log|z| - y \operatorname{amp} z - x], \qquad \ldots(2.2)$$

where $z = x + iy$, M is a positive constant independent of z and $g = R_e(\nu)$.

Let $F(\xi)$ denote the factors of the integrand in (1.1) which contain ξ and $\xi + \eta$ (η is considered to be constant) only. Applying the formula (2.1), we have

$$|F(\xi)| = \left| \frac{\prod_{j=1}^{p_1} \Gamma(a_j + A_j\eta + A_j\xi) \prod_{j=m_1+1}^{p_1} \sin(1 - a_j - A_j\eta - A_j\xi)\pi}{\prod_{j=1}^{q_1} \Gamma(b_j + B_j\eta + B_j\xi) \prod_{j=1}^{q_2} \Gamma(1 - d_j + D_j\xi)} \right.$$

$$\left. \times \frac{\prod_{j=1}^{p_2} \Gamma(1 - c_j + C_j\xi) \prod_{j=m_2+1}^{p_2} \sin(c_j - C_j\xi)\pi}{\prod_{j=1}^{n_2} \sin(d_j - D_j\xi)\pi\,(\pi)^{p_1 - p_2 - m_1 - m_2 - n_2}} x^{\xi} \right| \qquad \ldots(2.3)$$

Assuming ξ to be large and $|\arg \xi| < \pi - \delta$ if $\xi = R_1 e^{i\theta}$ so that amp $\xi = 0$, $|\xi| = R_1$, $x = r_1 e^{i\phi}$ and $\xi = \sigma + it$, then by virtue of (2.2), we have

$$|F(\xi)| \leqslant MR_1^{\rho} \exp\left[\left(\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j\right)\{(\log R_1 - 1)\sigma - t\theta\}\right.$$

$$\left. + |t|\left(\sum_{m_1+1}^{p_1} A_j + \sum_{m_2+1}^{p_2} C_j - \sum_1^{n_2} D_j\right)\pi + \sigma \log r_1 - t\phi\right].$$

The last expression, after little simplification, assumes the following form:

$$|F(\xi)| \leqslant MR_1^{\rho} \exp\left[\left(\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j\right)\{(\log R_1 - 1)\sigma\right.$$

$$+ |t|(\pi \pm \theta)\} + \sigma \log r_1 - |t|\left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_{n_2+1}^{q_2} D_j\right.$$

$$\left.\left. - \sum_1^{q_1} B_j\right)\pi \pm t\phi\right] \qquad \ldots(2.4)$$

where M and ρ are the numbers independent of ξ, $\sigma = R_1 \cos\theta$ and $t = R_1 \sin\theta_2$

Now, we consider the integral taken round that part of the circle $|\xi| = R_1$, where R_1 is large, which lies to the right of imaginary axis in the ξ-plane:

(*a*) Let us assume that

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j < 0,$$

if $0 \leqslant \theta \leqslant \pi/4$, so that $\sigma = R_1 \cos\theta \geqslant R_1/\sqrt{2}$ and further if

$$|\arg x| < \left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi,$$

then

$$|F(\xi)| \leqslant MR_1^{\rho} \exp\left[\left(\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j\right)\left\{(\log R_1 - 1)\frac{R_1}{\sqrt{2}}\right\}\right.$$

$$\left. + R_1 |\log r_1|\right], \qquad \ldots(2.5)$$

n being any finite number, it therefore follows that $|\xi^n F(\xi)|$ tends uniformly to zero as $R_1 \to \infty$.

(*b*) If

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j < 0 \quad \text{and if} \quad \frac{\pi}{4} \leqslant |\theta| \leqslant \frac{\pi}{2}$$

and R_1 is large enough so that $|t| = R_1 \sin|\theta| \geqslant \dfrac{R_1}{\sqrt{2}}$ and further if

$$|\arg x| < \left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi,$$

then

$$|F(\xi)| \leqslant MR_1^{\rho} \exp\left[-\frac{R_1}{\sqrt{2}}\left\{\left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi \pm \phi\right\}\right]. \qquad (2.6)$$

Therefore, n being any finite number, $|\xi^n F(\xi)|$ tends uniformly to zero as $R_1 \to \infty$.

(*c*) Again, if

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j = 0, \quad \text{and if} \quad 0 \leqslant \theta \leqslant \frac{\pi}{4}$$

so that $\sigma = R_1 \cos\theta \geqslant R_1/\sqrt{2}$

and if

$$|\arg x| < \left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi,$$

and

r_1 must be taken less than unity so that $-\sigma \log \frac{1}{r_1} \leqslant 0$,

then

$$|F(\xi)| \leqslant MR_1{}^{\rho} \exp\left[-\frac{R_1}{\sqrt{2}} \log \frac{1}{r_1}\right]. \qquad \ldots(2.7)$$

Hence $|\xi^n F(\xi)|$ tends uniformly to zero in this case also when $R_1 \to \infty$.

(*d*) Next, if we assume

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j = 0$$

and if

$$\frac{\pi}{4} < |\theta| < \frac{\pi}{2}$$

so that

$$|t| = R_1 \sin \theta \geqslant R_1/\sqrt{2}$$

and if

$$|\arg x| < \left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi,$$

and r_1 be taken less than unity so that

$$|F(\xi)| \leqslant MR_1{}^{\rho} \exp\left[-\frac{R_1}{\sqrt{2}}\left\{\left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi \pm \phi\right\}\right]. \qquad \ldots(2.8)$$

Thus, n being any finite number, $|\xi^n F(\xi)|$, in this case also, tends uniformly to zero as $R_1 \to \infty$.

From the above cases (*a*) to (*d*), we arrive at the conclusion that if

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j < 0$$

and

$$|\arg x| < \left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi,$$

then the integral round the semi-circle tends to zero when R_1 tends to infinity.

Also if

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j = 0$$

and

$$|\arg x| < \left(\sum_1^{m_1} A_j + \sum_1^{m_2} C_j - \sum_1^{q_1} B_j - \sum_{n_2+1}^{q_2} D_j\right)\pi,$$

then the integral round the semi-circle tends to zero when R_1 tends to infinity provided that $|x| = r_1 < 1$.

Now, we consider the integral in the η-plane, taken round the closed contour L_2, consisting of the imaginary axis from $-iR_2$ to $+iR_2$, where R_2 is large and that part of the semi-circle $|\eta| = R_2$ which lies to the right of the imaginary axis, R_2 being so chosen that the circle always passes between the poles of the integrand.

The discussion in the η-plane, considering ξ to be constant, is similar to that given above, and therefore for lack of space, we give below only the results.

(*e*) If

$$\sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_3} F_j < 0$$

and

$$|\arg y| < \left(\sum_1^{m_1} A_j + \sum_1^{m_3} E_j - \sum_1^{q_1} B_j - \sum_{n_3+1}^{q_3} F_j\right)\pi,$$

then the integral round the semi-circle tends to zero when R_2 tends to infinity.

(*f*) if

$$\sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_3} F_j = 0$$

and

$$|\arg y| < \left(\sum_1^{m_1} A_j + \sum_1^{m_3} E_j - \sum_1^{q_1} B_j - \sum_{n_3+1}^{q_3} F_j\right)\pi,$$

then the integral round the semi-circle tends to zero when R_2 tends to infinity provided that $|y| = r_2 < 1$.

Section II

In this section, we consider the integral in the ξ-plane taken round the closed contour L_3. The closed contour L_3 starts from $-ie^{-i\psi_1} R_3$ to $+ie^{i\psi_1} R_3$, where $0 < \psi_1 < \pi/2$ and R_3 is large. The remaining part of the contour consists of that part of the circle $|\xi| = R_3$ to the left of the imaginary axis from $ie^{i\psi_1} R_3$ to $-ie^{-i\psi_1} R_3$,

R_3 being so chosen that the circle always passes between the poles of the integrand. For simplicity we replace ξ by $-\xi$ so that when ξ is large

$$| F(-\xi) | \leqslant M_1 R_3^{\rho_1} \exp \Big[\Big(\sum_1^{q_1} B_j + \sum_1^{q_2} D_j - \sum_1^{p_2} C_j - \sum_1^{p_1} A_j \Big) \times \{ \sigma (\log R_3 - 1) + | t | (\pi \pm \theta) \} - \sigma \log r_1 + | t | \Big\{ \Big(\sum_{m_1+1}^{p_1} A_j + \sum_{m_2+1}^{p_2} C_j - \sum_1^{n_2} D_j \Big) \pi \pm \phi \Big\} \Big], \quad ...(2.9)$$

where M_1 and ρ_1 are numbers independent of ξ and $\sigma = R_3 \cos \theta$ and $t = R_3 \sin \theta$.

Proceeding in a manner as in Section I, we obtain the following results from (2.9):

If

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} E_j - \sum_1^{q_2} D_j > 0$$

and

$$| \arg x | < \Big(\sum_{m_1+1}^{p_1} A_j + \sum_{m_2+1}^{p_2} C_j - \sum_1^{n_2} D_j \Big) \pi,$$

then the integral round the semi-circle tends to zero when $R_3 \to \infty$. Also, if

$$\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} E_j - \sum_1^{q_2} D_j > 0$$

and

$$| \arg x | < \Big(\sum_{m_1+1}^{p_1} A_j + \sum_{m_2+1}^{p_2} C_j - \sum_1^{n_2} D_j \Big) \pi,$$

then the integral round the semi-circle tends to zero, when R_3 tends to infinity provided that $| x | = r_1 > 1$.

In a similar way, we consider the integral in the η-plane round the closed contour L_4. The closed contour L_4 starts from $-ie^{-i\psi_2} R_4$ to $+ie^{i\psi_2} R_4$, where $0 < \psi_2 < \pi/2$ and R_4 is large. The remaining part of the contour consists of that part of the circle $| \eta | = R_4$ to the left of the imaginary axis from $ie^{i\psi_2} R_4$ to $-ie^{i\psi_2} R_4$, R_4 being so chosen that the circle always passes between the poles of the integrand. We mention below the results only:

if

$$\sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_3} F_j > 0$$

and

$$| \arg y | < \Big(\sum_{m_1+1}^{p_1} A_j + \sum_{m_3+1}^{p_3} E_j - \sum_1^{n_3} F_j \Big) \pi,$$

then the integral round the semi-circle tends to zero when R_4 tends to infinity.

If

$$\sum_{1}^{p_1} A_j + \sum_{1}^{p_3} E_j - \sum_{1}^{q_1} B_j - \sum_{1}^{q_3} F_j = 0$$

and

$$|\arg y| < \left(\sum_{m_1+1}^{p_1} A_j + \sum_{m_3+1}^{p_3} E_j - \sum_{1}^{n_3} F_j \right) \pi,$$

then the integral round the semi-circle tends to zero when R_4 tends to infinity provided that $|y| = r_2 > 1$.

Section III

Now, the integral (1.1) is taken up the contour from $-i\infty$ to $+i\infty$ when $|t| = R_1$ is large, $\sigma = 0$, $\theta = \pm \pi/2$, then from (2.4) we get

$$|F(\xi)| \leqslant MR_1^{\rho} \exp \Big[-R_1 \Big\{ \Big(\sum_{1}^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_{1}^{q_1} B_j + \sum_{1}^{m_2} C_j - \sum_{m_2+1}^{p_2} C_j + \sum_{1}^{n_2} D_j - \sum_{n_2+1}^{q_2} D_j \Big) \frac{\pi}{2} \pm \phi \Big\} \Big] , \quad \ldots(2.10)$$

and thus it follows that the integral is an analytic function of x, provided that

$$|\arg x| < \left(\sum_{1}^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_{1}^{q_1} B_j + \sum_{1}^{m_2} C_j - \sum_{m_2+1}^{p_2} C_j + \sum_{1}^{n_2} D_j - \sum_{n_2+1}^{q_2} D_j \right) \frac{\pi}{2} ,$$

and

$$\lambda = \left(\sum_{1}^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_{1}^{q_1} B_j + \sum_{1}^{m_2} C_j - \sum_{m_2+1}^{p_2} C_j + \sum_{1}^{n_2} D_j - \sum_{n_2+1}^{q_2} D_j \right) > 0. \quad \ldots(2.11)$$

The above result holds good for all values of $|x| = r_1$.

Similarly, the integral is an analytic function of y provided that

$$|\arg y| < \left(\sum_{1}^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_{1}^{q_2} B_j + \sum_{1}^{m_3} E_j - \sum_{m_3+1}^{p_3} E_j + \sum_{1}^{n_3} F_j - \sum_{n_3+1}^{q_3} F_j \right) \frac{\pi}{2},$$

and

$$\mu = \left(\sum_{1}^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j - \sum_{1}^{q_1} B_j + \sum_{1}^{m_3} E_j - \sum_{m_3+1}^{p_3} E_j \sum_{1}^{n_3} + F_j - \sum_{n_3+1}^{q_3} F_j \right) > 0. \quad \ldots(2.12)$$

Thus, we find that the integral (1.1) is an analytic function of x and y provided that

$$\left.\begin{array}{l}\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j < 0, \\ \sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_3} F_j > 0, \\ |\arg x| < \tfrac{1}{2}\lambda\pi, \qquad |\arg y| < \tfrac{1}{2}\mu\pi;\end{array}\right\} \quad \text{(i)}$$

$$\left.\begin{array}{l}\sum_1^{p_1} A_j + \sum_1^{p_2} C_j - \sum_1^{q_1} B_j - \sum_1^{q_2} D_j = 0, \\ \sum_1^{p_1} A_j + \sum_1^{p_3} E_j - \sum_1^{q_1} B_j - \sum_1^{q_2} F_j = 0, \\ |\arg x| < \tfrac{1}{2}\lambda\pi, \quad |\arg y| < \tfrac{1}{2}\mu\pi, \\ |x| < R_1 \leqslant 1, \quad |y| < R_2 \leqslant 1,\end{array}\right\} \quad \text{(ii)}$$

where λ and μ are the same as defined in (2.11) and (2.12) respectively.

3. A Multiplication Theorem

Assume that the conditions (i) and (ii) as stated above hold and further let $|1 - \lambda_1^{-1/D_1}| < 1$ and $|1 - \mu_1^{-1/F_1}| < 1$, then

$$H\left[\begin{matrix}\lambda_1 x \\ \mu_1 y\end{matrix}\left|\begin{matrix}(a, A); (b, B) \\ (c, C); (d, D) \\ (e, E); (f, F)\end{matrix}\right.\right] = \lambda_1^{d_1/D_1} \mu_1^{f_1/F_1} \sum_{r,k=0}^{\infty} \frac{(1 - \lambda_1^{1/D_1})^r \; (1 - \mu_1^{1/F_1})^k}{r! \; k!}$$

$$\times H\left[\begin{matrix}x \\ y\end{matrix}\left|\begin{matrix}(a, A); (b, B) \\ (c, C); (d_1 + r, D_1), \{(d_2, D_2)_{q_2}\} \\ (e, E); (f_1 + k, F_1), \{(f_2, F_2)_{q_3}\}\end{matrix}\right.\right] \quad \ldots(3.1)$$

where $\{(d_2, D_2)_{q_2}\}$ stands for $(d_2, D_2), \ldots, (d_{q_2}, D_{q_2})$ and so on.

Proof: The proof is easy to visualize and is based on the well-known property ${}_1F_0[a; - ; x] = (1 - x)^{-a}$.

Alternatively if we let $|1 - \lambda_1^{-1/C_1}| < 1$ and $|1 - \mu_1^{-1/E_1}| < 1$, we shall have

$$H\left[\begin{matrix}\lambda_1 x \\ \mu_1 y,\end{matrix}\left|\begin{matrix}(a, A); (b, B) \\ (c, C); (d, D) \\ (e, E); (f, F)\end{matrix}\right.\right] = \lambda_1^{(c_1-1)/C_1} \mu_1^{(e_1-1)/E_1}$$

$$\times \sum_{r,k=0}^{\infty} \frac{(1 - \lambda_1^{-1/C_1})^r \; (1 - \mu_1^{-1/E_1})^k}{r! \; k!} \times$$ [continued on p. 112]

$$\times H \left[\begin{array}{c|r} x & (a, A); (b, E) \\ y & (c_1 - r, C_1), \{(c_2, C_2)_p\}; (d, D) \\ & (e_1 - k, E_1), \{(e_2 E_2)_{p_2}\}; (f, F) \end{array} \right]. \quad ...(3.2)$$

A typical particular case of (3.1) is a result of Saxena (1973) that is obtained on setting $m_1 = p_1 = q_1 = p_3 = q_3 = n_3 = 0 = m_3$.

4. An Inter-relation

Let the conditions (i) and (ii) of Section III of § 2 be satisfied in addition to the condition

$$\text{Re}\left(1 + l + \gamma \min \frac{d_i}{D_i}\right) > 0, \quad (i = 1, \ldots, n_2). \tag{i}$$

Further let $J^{\rho}_{\beta}(z)$ be the Maitland function (Wright 1933, 1934) satisfying the integral property

$$\int_0^{\infty} z^l J_{\beta}{}^{\rho}(z)\, dz = \frac{\Gamma(1+\rho)}{\Gamma(1+\beta-\rho-l\rho)}, \quad \text{Re}(l)) > 0, 0 < \rho \leqslant 1, \tag{ii}$$

and the recursion relation

$$\rho z \phi_{\beta+\rho} = \phi_{\beta-1} + (1-\beta)\phi_{\beta} \tag{iii}$$

where

$$J^{\rho}_{\beta-1}(-z) = \phi_{\beta}(\rho, \beta, z) = \sum_{r=0}^{\infty} \frac{z^r}{r!\ \Gamma(\beta + \rho r)};$$

then, we have the inter-relation

$$(C_1 c_{p_2} - C_1 + C_{p_2} - C_{p_2} c_1) H \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= C_{p_2} H \left[\begin{array}{c|r} x & (a, A) \;;\; (b, B) \\ y & (c_1 - 1, C_1), \{(c_2, C_2)_{p_2}\} \;;\; (d, D) \\ & (e, E) \;;\; (f, F) \end{array} \right]$$

$$+ C_1 H \left[\begin{array}{c|r} x & (a, A) \;;\; (b, B) \\ y & \{(c_1 - C_1)_{p_2-1}\}, (c_{p_2} - 1, C_{p_2}) \;;\; (d, D) \\ & (e, E) \;;\; (f, F) \end{array} \right]. \quad ...(4.1)$$

PROOF: To prove, multiply (iii) throughout by $z^l H\left[\begin{smallmatrix} xz^\gamma \\ y \end{smallmatrix}\right]$, integrate with respect to z from 0 to ∞, using (ii) and change of order of integration, which is justified by de la Vallee Poussin's theorem (Bromwich 1931), since the integrals involved in the process are absolutely convergent in view of the conditions stated, we shall have

$$-\rho\, H^{m_1,\,0\,;\,m_2+1,\,n_2\,;\,m_3,\,n_3}_{p_1,\,q_1\,:\,p_2+2,\,q_2\,;\,p_3,\,q_3}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (a,A);\,(b,B) \\ (-l-1,\gamma),\,(c,C);\,(\beta-\rho-\rho l,\rho\gamma),\,(d,D) \\ (e,E);\,(f,F) \end{matrix}\right]$$

$$= H^{m_1,\,0\,;\,m_2+1,\,n_2\,;\,m_3,\,n_3}_{p_1,\,q_1\,;\,p_2+2,\,q_2\,;\,p_3,\,q_3}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (a,A);\,(b,B) \\ (-l,\gamma),\,(c,C);\,(\beta-\rho-\rho l-1,\rho\gamma),\,(d,D) \\ (e,E);\,(f,F) \end{matrix}\right]$$

$$+(1-\beta)\, H^{m_1,\,0\,;\,m_2+1,\,n_2\,;\,m_3,\,n_3}_{p_1,\,q_1\,;\,p_2+2,\,q_2\,;\,p_3,\,q_3}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (a,A);\,(b,B) \\ (-l,\gamma),\,(c,C);\,(\beta-\rho-\rho l,\rho\gamma),\,(d,D) \\ (e,E);\,(f,F) \end{matrix}\right] \quad ...(4.2)$$

On making suitable adjustment of parameters and reduction in the order of indices, *viz.*, replacing l by $-c_1$, γ by C_1, β by c_{p_2}, ρ by C_{p_2}, m_2+1 by m_2 and p_2+2 by p_2, it will lead us to the desired result (4.1).

An interesting corollary of (4.1) is yet another result of Saxena (1973) which follows on setting $p_1 = q_1 = m_1 = p_3 = q_3 = m_3 = n_3 = 0$.

In view of the cases of reducibility and the asymptotic properties as discussed in Munot and Kalla (1971) and Dubey and Sharma (1972), it is easy to conclude that $H\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ not only includes the Fox's H-function or the product of two H-functions as its special cases, but it also incorporates most of the commonly used functions of one and two variables such as generalized G or S-function of two variables which, in turn, yield the product of two G-function or a Meijer's G-function or a MacRobert's E-function, the Kampe de Ferriet's double hypergeometric functions, the Appell functions F_1, F_2, F_3 and F_4, the Whittaker function of two variables, Maitland's generalized hypergeometric function, etc., on suitably adjusting the parameters and reducing the order. Evidently, this will enable us to obtain several useful corrollaries of our results (3.1) and (4.1).

5. EXTENSION OF H-FUNCTION TO $\dot{H}$

It was the idea of symmetry which the H-function of two variables lacked and which was an inherent characteristic property of the well-known Meijer's

G-function and Fox's H-function, that we were motivated to define what we refer to as the $\overset{*}{H}$-function. We make the definition

$$\overset{*}{H}{}^{m_1, n_1; m_2, n_2; m_3, n_3}_{p_1, q_1; p_2, q_2; p_3, q_3}\left[\begin{array}{c|c} x & (a_{p_1}; \alpha_{p_1}, A_{p_1}); (b_{q_1}; \beta_{q_1}, B_{q_1}) \\ & (C_{p_2}, C_{p_2}); (d_{q_2}, D_{q_2}) \\ y & (e_{p_3}, E_{p_3}); (f_{q_3}, F_{q_3}) \end{array}\right] = \overset{*}{H}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} U(\xi, \eta)\, V(\xi)\, W(\eta)\, d\xi\, d\eta. \qquad \ldots(5.1)$$

The functions $U(\xi, \eta)$, $V(\xi)$ and $W(\eta)$ are defined as follows :

$$\left.\begin{aligned} U(\xi, \eta) &= \frac{\prod_{j=1}^{m_1} \Gamma(a_j + \alpha_j \xi + A_j \eta) \prod_{j=1}^{n_1} \Gamma(1 - b_j - \beta_j \xi - B_j \eta)}{\prod_{j=m_1+1}^{p_1} \Gamma(1 - a_j - \alpha_j \xi - A_j \eta) \prod_{j=n_1+1}^{q_1} \Gamma(b_j + \beta_j \xi + B_j \eta)} \\ V(\xi) &= \frac{\prod_{j=1}^{m_2} \Gamma(1 - c_j + C_j \xi) \prod_{j=1}^{n_2} \Gamma(d_j - D_j \xi)}{\prod_{j=m_2+1}^{p_2} \Gamma(c_j - C_j \xi) \prod_{j=n_2+1}^{q_2} \Gamma(1 - d_j + D_j \xi)}\, x^{\xi} \\ W(\eta) &= \frac{\prod_{j=1}^{m_3} \Gamma(1 - e_j + E_j \eta) \prod_{j=1}^{n_3} \Gamma(f_j - F_j \eta)}{\prod_{j=m_3+1}^{p_3} \Gamma(e_j - E_j \eta) \prod_{j=n_3+1}^{q_3} \Gamma(1 - f_j + F_j \eta)}\, y^{\eta}. \end{aligned}\right\} \qquad \ldots(5 \cdot 2)$$

It shall be understood that the sequence $(a_{p_1}; \alpha_{p_1}, A_{p_1})$ stands for the set of p_1 parameters $(a_1; \alpha_1, A_1), \ldots, (a_{p_1}; \alpha_{p_1}, A_{p_1})$ with similar interpretations for $(1 - a_{p_1}; \alpha_{p_1}, A_{p_1}), \ldots$, etc. Furthermore, as in the definition of H-function, we assume that x and y are not equal to zero and place similar restrictions on the meaning and scope of a's ...; α's, A's... etc. It shall also be assumed that an empty product will be interpreted as unity and that the non-negative integers p_i, q_i, m_i, n_i $(i = 1, 2, 3)$, satisfy the inequalities $p_i \geqslant m_i \geqslant 0$, $q_i \geqslant n_i \geqslant 0$.

The contour L_1 in the ξ-plane runs from $-i\infty$ to $+i\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(d_j - D_j\xi)$, $J = 1, \ldots, n_2$; $\Gamma(1 - b_j - B_j\xi - B_j\eta)$, $j = 1, \ldots, n_1$, lie to the right and the poles of $\Gamma(1 - c_j + C_j\xi)$, $j = 1, \ldots, m_2$, and $\Gamma(a_j + \alpha_j\xi + A_j\eta)$, $j = 1, \ldots, m_1$, lie to the left of the contour. Similarly, the contour L_2 in the η-plane runs from $-i\infty$ to $+i\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$, $j = 1, \ldots, n_3$, and $\Gamma(1 - b_j - \beta_j\xi - B_j\eta)$, $j = 1, \ldots, n_1$; lie to the right and the poles of $\Gamma(1 - e_j + E_j\eta)$, $j = 1, \ldots, m_3$, and $\Gamma(a_j + \alpha_j\xi + A_j\eta)$, $j = 1, \ldots, m_1$; lie to the left of the contour.

In analogy with the procedure described in § 2, we conclude straightaway that the function $\overset{*}{H}\begin{bmatrix} x \\ y \end{bmatrix}$ is an analytic function of x and y in the entire plane provided it satisfies the following sets of convergence conditions:

$$\left.\begin{aligned}
&\sum_{j=1}^{p_1} a_j + \sum_{j=1}^{p_2} C_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} D_j < 0,\\
&\sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j < 0\\
&|\arg x| < \tfrac{1}{2}\lambda_1 \pi \quad \text{and} \quad |\arg y| < \tfrac{1}{2}\mu_1 \pi;\\
&\text{where } \lambda_1 \text{ and } \mu_1 \text{ are given by}\\
&\lambda_1 = \Big(\sum_{1}^{m_1} a_j - \sum_{m_1+1}^{p_1} a_j + \sum_{1}^{m_2} C_j\\
&\qquad - \sum_{m_2+1}^{p_2} C_j + \sum_{1}^{n_1} \beta_j - \sum_{n_1+1}^{q_1} \beta_j\\
&\qquad + \sum_{1}^{n_2} D_j - \sum_{n_2+1}^{q_2} D_j \Big) > 0\\
&\mu_1 = \Big(\sum_{1}^{m_1} A_j - \sum_{m_1+1}^{p_1} A_j + \sum_{1}^{m_3} E_j\\
&\qquad - \sum_{m_3+1}^{p_3} E_j + \sum_{1}^{n_1} B_j - \sum_{n_1+1}^{q_1} B_j\\
&\qquad + \sum_{1}^{n_3} F_j - \sum_{n_3+1}^{q_3} F_j \Big) > 0
\end{aligned}\right\} \qquad \text{(i)}$$

$$\left.\begin{aligned}
&\sum_{j=1}^{p_1} a_j + \sum_{j=1}^{p_2} C_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} D_j = 0,\\
&\sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j = 0,\\
&|\arg x| < \tfrac{1}{2}\lambda_1 \pi \quad \text{and} \quad |\arg y| < \tfrac{1}{2}\mu_1 \pi,\\
&\qquad |x| < R_1 \leqslant 1, \quad |y| < R_2 \leqslant 1
\end{aligned}\right\} \qquad \text{(ii)}$$

where λ_1 and μ_1 are the same as defined in (i) above.

While concluding, we shall like to remark that the cases of reducibility and other properties of the $\overset{*}{H}$-function could be developed on parallel lines to those of Agarwal's G-function of two variables. Thus in view of the remark in § 4, whereas on the one hand we shall obtain several specialized forms of our theorems (3.1) and (4.1); on the other hand the whole range of hypergeometric function theory could be viewed from a more general standpoint. This in effect will amount to obtaining various new results in terms of the $\overset{*}{H}$-function.

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