COSMOLOGICAL SOLUTION OF EINSTEIN'S EQUATIONS IN GENERAL RELATIVITY

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A model is proposed in which the central pressure and central density are finite.

1. INTRODUCTION

Krori *et al.* (1974) found an internal solution for a spherically symmetric matter distribution taking

$$\rho c^2 + 3p = f(r).$$

Heie in the present paper, we have attempted a solution of the Einstein's field equations taking cosmological constant in the solution. The constant k_1 is any arbitrary positive value of the order of Λ, k_3 is negative and k_2 is positive lying between 0 and 1. With positive value of Λ, it is found that radius increases and consequently density at the centie increases to counteract this expansion.

2. FIELD EQUATIONS AND THEIR SOLUTIONS

The line element exhibiting spherically symmetric matter distribution (Møller 1952, p. 329),

$$ds^2 = a(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - b(r)\,c^2\,dt^2. \qquad \ldots(2.1)$$

The equations to be satisfied are

$$p' + (\rho c^2 + p)\frac{b'}{2b} = 0 \qquad \ldots(2.2)$$

$$\frac{b'}{abr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \Lambda = 8\pi p \qquad \ldots(2.3)$$

$$\frac{a'}{a^2 r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \Lambda = 8\pi\rho c^2 \qquad \ldots(2.4)$$

where prime denotes differentiation with reference to r and Λ is the cosmological constant. Now, we assume the solution

$$b = \tfrac{1}{2} k_1 r^2 + k_2 \quad \ldots(2.5)$$

where k_1 and k_2 are constants of integration. Following Krori *et al.* (1974), we have,

$$p = \frac{1}{16\pi}\left(\frac{b'}{abr}\right) + k_3 \quad \ldots(2.6)$$

where k_3 is a constant of integration.

From equation (2.3)

$$a = \frac{k_1 r^2 + k_2}{(1 + 8\pi k_3 r^2 - r^2 \Lambda)(\frac{1}{2} k_1 r^2 + k_2)}. \quad \ldots(2.7)$$

Hence from equation (2.6)

$$p = \frac{1}{16\pi}\left(\frac{1 + 8\pi k_3 r^2 - r^2 \Lambda}{r^2 + \dfrac{k_2}{k_1}}\right) + k_3 \quad \ldots(2.8)$$

and from (2.4)

$$8\pi\rho c^2 = \frac{\dfrac{k_1{}^2}{2}(\Lambda - 24\pi k_3) r^4 + \frac{1}{2} k_1 r^2 (k_1 + 3k_2\Lambda - 56\pi k_2 k_3) + k_2(\frac{3}{2} k_1 + 2k_2\Lambda - 24\pi k_2 k_3)}{(k_1 r^2 + k_2)^2} \quad \ldots(2.9)$$

The boundary is given by

$$r_1 = \left\{\frac{k_1 + 16\pi k_2 k_3}{k_1(\Lambda - 24\pi k_3)}\right\}^{1/2} \quad \ldots(2\cdot10)$$

At the boundary, since $a(r_1)\, b(r_1) = 1$, we have

$$(k_1 + \Lambda + 8\pi k_3) r_1{}^2 = (1 - k_2). \quad \ldots(2.11)$$

Putting the value of r_1 in equation (2.11) and simplifying, we have

$$k_1 = -16\pi k_3. \quad \ldots(2.12)$$

Thus if k_1 is positive and of the order of Λ, k_3 will be negative, hence from (2.11) k_2 lies between 0 and 1.

Now,

$$b(r_1) = 1 - \frac{2m}{r_1} - \frac{1}{3}\Lambda r_1{}^2$$

at the exterior gives

$$\frac{2m}{r_1} = \frac{1}{2} k_1 r_1{}^2 + \frac{2}{3} r_1{}^2 (\Lambda - 12\pi k_3). \quad \ldots(2.13)$$

Since k_1 is any arbitrary positive value of the order of Λ and since k_3 is negative, $2m/r_1$ is positive.

Equation (2.13) gives using the value of k_3 from equation (2.12)

$$k_1 = \frac{2m}{r_1^3} - \frac{2}{3}\Lambda \qquad \text{...(2.14)}$$

where Λ (lower limit) $\simeq 9 \cdot 3 \times 10^{-58}\,\text{cm}^{-2}$ (Tolman 1964, p. 345). Since k_1 is considered a positive value, hence m/r_1^3 must be greater than $\frac{1}{3}\Lambda$.

At $r = 0$, we have

$$8\pi\rho_0 c^2 = \frac{3k_1}{2k_2} + 2(\Lambda - 12\pi k_3) \qquad \text{...(2.15)}$$

$$8\pi p_0 = \frac{k_1}{2k_2} + 8\pi k_3\,. \qquad \text{...(2.16)}$$

From above it is found that $\rho_0 c^2$ and p_0 are positive and $\rho_0 c^2 > 3p_0$. Also we have

$$\frac{\rho_0 c^2}{p_0} = 3\left(\frac{1+k_2}{1-k_2}\right) + \frac{4k_2\Lambda}{\left(\frac{2m}{r_1^3} - \frac{2}{3}\Lambda\right)(1-k_2)}\,. \qquad \text{...(2.17)}$$

The equation (2.10) can be simplified to as

$$r_1 = \left(\frac{1-k_2}{\Lambda + \frac{3}{2}k_1}\right)^{1/2} \qquad \text{...(2.18)}$$

and it is real since k_1 is positive and k_2 lies between 0 to 1.

Since Λ is taken as a positive value, the mass density at the centre is slightly increased [eqn. (2.15)] and this is necessary to counteract any expansion [eqn. (2.18)] that arises due to the inclusion of cosmological term Λ in the solution.

References

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