ON THE STEPANOV-BOUNDED SOLUTIONS OF CERTAIN ABSTRACT DIFFERENTIAL EQUATIONS

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In a Hilbert space, the Stepanov-bounded solutions of certain first-order non-homogeneous linear differential equations with a Stepanov-almost periodic forcing function are almost periodic.

§ 1. Suppose H is a Hilbert space and J is the interval $-\infty < t < \infty$. A function $f \in L^p_{loc}(J; H)$ with $1 \leq p < \infty$ is said to be Stepanov-bounded or S^p-bounded on J if

$$\| f \|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \| f(s) \|^p \, ds \right]^{1/p} < \infty \qquad \ldots(1.1)$$

(for the definitions of almost periodicity and S^p-almost periodicity (*see* Amerio and Prouse 1971, pp. 3 and 77).

We consider the first-order linear differential equation

$$u'(t) = Au(t) + f(t) \quad \text{on } J \qquad \ldots(1.2)$$

where $f : J \to H$ is an S^1-almost periodic continuous function, and A is a completely continuous normal operator on H into itself.

Our first result is as follows.

Theorem 1—If H is separable, then any S^1-bounded solution of (1.2) is almost periodic from J to H.

§ 2. We shall require the following result.

Lemma—If $g : J \to H$ is a continuously differentiable function, with g and g' both being S^1-bounded on J, then g is bounded on J.

PROOF: For an arbitrary but fixed $t \in J$, there exists at least one point $r_t \in [t-1, t]$ such that

$$\| g(r_t) \| = \inf_{t-1 \leq s \leq t} \| g(s) \|. \qquad \ldots(2.1)$$

Consequently, we have

$$\| g(r_t) \| \leq \int_{t-1}^{t} \| g(s) \| \, ds \leq \| g \|_{S^1}, \text{ by } (1.1). \qquad \ldots(2.2)$$

So we have, by the S^1-boundedness of g',

$$\| g(t) \| = \| g(r_t) + \int_{r_t}^{t} g'(s) \, ds \|$$

$$\leq \| g(r_t) \| + \int_{r_t}^{t} \| g'(s) \| \, ds$$

$$\leq \| g \|_{S^1} + \| g_1' \|_{S^1} \qquad \ldots(2.3)$$

§ 3. *Proof of Theorem* 1—Since f is S^1-almost periodic from J to H, it is S^1-bounded on J. So $u' = Au + f$ is S^1-bounded on J. Consequently, by our Lemma, u is bounded on J.

By (1.2), we have the representation

$$u(t) = u(0) + \int_0^t Au(s) \, ds + \int_0^t f(s) \, ds \text{ on } J. \qquad \ldots(3.1)$$

If $t_2 > t_1$, then

$$\left\| \int_{t_1}^{t_2} Au(s) \, ds \right\| \leq \| A \| \cdot \sup_{t \in J} \| u(t) \| \cdot (t_2 - t_1). \qquad \ldots(3.2)$$

Hence $\int_0^t Au(s) \, ds$ is uniformly continuous on J. Further, by Theorem 8 (p. 79) of Amerio-Prouse (1971) $\int_0^t f(s) \, ds$ is uniformly continuous on J. Thus it follows that u is uniformly continuous on J.

Now consider a sequence $\{\psi_n(t)\}_{n=1}^{\infty}$ of continuous non-negative functions on J such that

$$\psi_n(t) = 0 \text{ for } |t| \geq n^{-1}, \int_{-n^{-1}}^{n^{-1}} \psi_n(t) \, dt = 1. \qquad \ldots(3.3)$$

The convolution between u and ψ_n is defined by

$$(u * \psi_n)(t) = \int_J u(t-s) \psi_n(s) \, ds = \int_J u(s) \psi_n(t-s) \, ds \text{ on } J. \qquad \ldots(3.4)$$

From (1.2), it follows that

$$(u * \psi_n)'(t) = A(u * \psi_n)(t) + (f * \psi_n)(t) \text{ on } J. \qquad \ldots(3.5)$$

Then we have

$$\sup_{t \in J} \| (u * \psi_n)(t) \| \leq \sup_{t \in J} \| u(t) \| . \qquad \ldots(3.6)$$

Also, we can show that $f * \psi_n$ is almost periodic from J to H [see the proof of Theorem 7 (p. 78) of Amerio-Prouse 1971]. Hence, by Theorem 1 of Cooke (1969) $u * \psi_n$ is almost periodic from J to H.

Now, by the uniform continuity of u on J, the sequence of convolutions $(u * \psi_n)(t)$ converges to $u(t)$ uniformly on J. Therefore, u is almost periodic from J to H. This completes the proof of the theorem.

§ 4. The proof of the following result parallels that of our Theorem 1. We have only to appeal to a theorem of Zaidman (1963) instead of Theorem 1 of Cooke (1969).

Theorem 2—If B is a self-adjoint bounded linear operator on a Hilbert space H into itself, and $f : J \to H$ is an S^1-almost periodic continuous function, then any S^1-bounded solution of the differential equation

$$u'(t) = Bu(t) + f(t) \quad \text{on } J \qquad \ldots(4.1)$$

is almost periodic from J to H.

§ 5. *Note*: Let $L(H, H)$ be the Banach space of all bounded linear operators on a Hilbert space H into itself, with the uniform operator topology. Suppose $A \in L(H, H)$ is a $1 - 1$ self-adjoint operator, with $0 \leq A \leq I =$ the identity operator on H, and $B \in L(H, H)$ commutes with A ($AB = BA$). Let $\{E_\lambda\}_{\lambda \in J}$ be the spectral family of A. We set $F_n = E_{1/n} - E_{1/(n+1)}$, for $n = 1, 2, \cdots$, and assume that

$$\sup_{t \in J} \| \exp(BF_n t) \|_{L(F_n(H), F_n(H))} < \infty \text{ for all } n \geq 1 \qquad \ldots(5.1)$$

$$\sum_{n=1}^{\infty} \| BF_n \|_{L(H,H)} < \infty. \qquad \ldots(5.2)$$

Then we have the following result.

Theorem 3—If $f : J \to H$ is an S^1-almost periodic continuous function, then any S^1-bounded solution of the differential equation

$$u'(t) = (A + B) u(t) + f(t) \quad \text{on } J \qquad \ldots(5.3)$$

is almost periodic from J to H.

The proof of this result is also similar to that of Theorem 1, but now we have to use another theorem of Zaidman (1972).

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