DOUBLE CENTRALISER ALGEBRA WITH THE STRICT TOPOLOGY

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Busby (1969) defined and studied the strict topology β on the double centraliser algebra $M(A)$ of a B^*-algebra A. In this paper we prove that if A has a countable approximate identity then $M(A)$, with the strict topology β is locally m–convex if and only if $M(A) = A$ and β is the norm topology. It is however shown by means of an example that this result cannot be extended to B^*-algebras with uncountable Abelian approximate identity.

Definitions and terminology—Let $C_b(S)$ be the B^*-algebra of all bounded complex valued, continuous functions on a locally compact, Hausdorff space S with the supnorm topology. Let $C_0(S)$ be the algebra of all functions in $C_b(S)$ which vanish at infinity, and let $C_b(S)_\beta$ denote $C_b(S)$ under the strict or β topology. Wang (1961) has proved that $C_b(S)$ is the double centraliser algebra of $C_0(S)$. Busby (1969) introduced a generalized notion of the strict topology. For a B^*-algebra A, he defined and studied the strict topology β on the double centraliser algebra $M(A)$ of A. The topology β on $M(A)$ is generated by the seminorms $\{\lambda_a, \rho_a : a \in A\}$ where $\lambda_a(x) = \| ax \|$ and $\rho_a(x) = \| xa \|$. A has an identity if and only if $M(A) = A$ and β is the norm topology.

Arens (1952) and Michael (1952) introduced and studied locally m-convex algebras. A subset U of a locally convex algebra E is said to be idempotent if $U.U \subset U$. E is said to be a locally multiplicatively convex (in short lmc) algebra if there exists a base of idempotent, convex neighbourhoods of zero. Following Cochran *et al.* (1970) an A-convex seminorm is a seminorm p for which, for each $x \in E$, there are constants M_x and N_x s.t. $p(xy) \leqslant M_x\, p(y)$ and $p(yx) \leqslant N_x p(y)$ for all $y \in E$. An A-convex algebra is a locally convex algebra whose topology is defined by a family of A-convex seminorms. For a detailed account of the theory of double centralisers on a B^*-algebra, we refer to Busby (1969) and for definition and concepts in the theory of topological vector spaces we refer to Köthe (1969).

Theorem 1—Let A be a B^*-algebra with a countable approximate identity. Then $M(A)_\beta$ is locally m-convex if and only if $M(A) = A$ and β is the norm topology.

PROOF: Assume that $M(A)_\beta$ is locally m-convex. Now, since A has a countable approximate identity, A possesses an Abelian countable approximate identity by Theorem 1 of Aarnes and Kadison (1969). Let B be the commutative B^*-algebra generated by this Abelian countable approximate identity in A. Then $B = C_0(S)$ for some locally compact, σ-compact, Hausdorff space S. By Corollary 2.4 of Sentilles (1970) we see that β on $M(A)$ is the same as the topology generated by the seminorms $\{\lambda_b, \rho_b : b \in B\}$ where $\lambda_b(x) = \| bx \|$ and $\rho_b(x) = \| xb \|$ for $x \in M(A)$ and $b \in B$. Since $M(A)$ is locally m-convex, so is $B_\beta = C_0(S_\beta)$. The proof of Theorem 3.2 Cochran (1971) works to show that the finest locally m-convex topology on $C_0(S)$ coarser than β is the compact open topology k. So $\beta = k$ on $C_0(S)$. Since S is locally compact, Hausdorff and σ-compact, $S_0(C_0(S), \beta)$ is metrisable and dense in $(C_b(S), \beta)$ which is complete. Thus S is compact. Therefore B has an identity. Hence A has an identity.

Whether or not it can be extended to B^*-algebras nith Abelian approximate identity can be answered in the negative as shown in the following example.

Example—Consider the space S_0 of all ordinals less than the first uncountable ordinal Ω, with the order topology. Let $A = C_0(S)$. Then $M(A) = C_b(S_0)$. In this case $A \neq M(A)$ and $\beta = k$. Therefore $M(A)_\beta$ is locally m-convex. Hence $M(A)_\beta$ is locally m-convex without A being equal to $M(A)$. However A possesses an Abelian approximate identity, given by $\{e_\lambda : \lambda \in S_0\}$ where $e_\lambda(t) = 1$ for $t \in [0, \lambda]$ and zero otherwise.

Theorem 2—If A is a B^*-algebra with an Abelian approximate identity, then $M(A)_\beta$ is A-convex.

PROOF: Suppose $\{e_\lambda : \lambda \in \Lambda\}$ is the given Abelian approximate identity for A and let B be the commutative Banach algebra generated by $\{e_\lambda : \lambda \in \Lambda\}$. Then by Corollary 2.4 (Sentilles 1970) the strict topology β on $M(A)$ is given by the neighbourhoods $\{V_a : a \in B\}$, where $V_a = \{x \in M(A) : \lambda_a(x) \leqslant 1,\ \rho_a(x) \leqslant 1\}$.

Let $a \in B$ and consider V_a. Now it is sufficient to show that for each $a \in B$, V_a is A-convex.

Let y be any element of V_a so that

$$\| y_1(a) \| \leqslant 1, \quad \| y_2(a) \| \leqslant 1.$$

If $x \in M(A)$ then

$$\| xya \| = \| x_1 y_1(a) \| \leqslant \| x_1 \| \| y_1(a) \| \leqslant \| x_1 \| = \| x \|.$$

Now

$$\| axy \| = \| y_2 x_2(a) \|.$$

Since B is commutative, the double centraliser algebra $M(B)$ of B is commutative. Also since xy restricted to B is in $M(B)$ and $a \in B$, we get

$$\| axy \| = \| y_2 x_2(a) \| = \| x_2 y_2(a) \| \leqslant \| x_2 \| \| y_2(a) \| \leqslant \| x \|.$$

Hence

$$xV_a \subset \| x \| V_a.$$

Similarly it can be shown that $V_a x \subset \| x \| V_a$.

Hence, V_a is A-convex. This completes the proof.

Corollary 1—If A is a B^*-algebra having a countable approximate identity then $M(A)_\beta$ is an A-convex algebra.

PROOF: In this case by Theorem 1 of Aarnes and Kadison (1969), A possesses an Abelian approximate identity.

Remark: Cochran *et al.* (1970) defined P-completeness for an A-convex algebra. They also proved that the algebra $C_b(R)$ of bounded, continuous functions on the real line R with the strict topology β is not P-complete. So our next theorem is a stronger version of this result in the sense that if $A = C_0(R)$ then $M(A)_\beta = C_b(R)_\beta$.

Theorem 3—Let A be a B^*-algebra with a countable approximate identity. Then $M(A)_\beta$ is quasi-barrelled or P-complete if and only if $M(A) = A$ and β is the norm topology.

PROOF: By Corollary 1 we get that $M(A)_\beta$ is an A-convex algebra. Furthermore Busby (1969) has proved that $M(A)_\beta$ is complete. A sequentially complete, quasi-barrelled algebra is barrelled. It has been proved by Cochran *et al.* (1970) that a barrelled A-convex algebra is locally m-convex. Also a P-complete A-convex algebra is locally m-convex (Cochran *et al.* 1970). So an application of Theorem 1 gives the required result.

Corollary 1—Let A be a B^*-algebra with a countable approximate identity. Then $M(A)_\beta$ is metrisable if and only if $M(A) = A$ and β is the norm topology.

Remark: If A has a countable approximate identity then $M(A)_\beta$ is a Mackey space (Taylor 1970) which is an A-convex algebra. Therefore we have a plenty of examples of spaces in which the Mackey topology coincides with the strongest A-convex topology compatible with duality, which is not locally m-convex in case A does not have an identity.

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