ON PLANE WAVE-LIKE SOLUTIONS OF NON-SYMMETRIC UNIFIED FIELD THEORIES*

by S. N. PANDEY**, *Department of Applied Sciences, M.M.M. Engineering College, Gorakhpur (U.P.)*

(*Received* 11 *July* 1975)

The purpose of the present paper is to obtain some plane wave-like solutions of the field equations of non-symmetric unified field theories given by Einstein (1951) Bonnor (1954) and Schrödinger (1950) corresponding to solutions obtained recently by Lal and Pandey (1974).

1. INTRODUCTION

The study of plane wave or plane wave-like solutions of the field equations in general relativity and those of the field equations in non-symmetric unified field theory has been made by various investigators (see, for example, Takeno 1958, 1961; Vaidya and Pandya 1960; Patel and Vaidya 1971). The field equations of these theories differ only in some region where there exists some other field besides gravitation. Therefore we shall confine ourselves to some region in which both gravitational and electromagnetic fields are present.

Furthermore, Lal and Pandey (1974) have obtained plane wave-like solutions of the field equations in general relativity in the space-time given by

$$\left.\begin{aligned} & ds^2 = -Adx^2 - Bdy^2 - (1-E)\,dz^2 - 2Edzdt + (1+E)\,dt^2 \\ & A = A(z,t), \qquad B = B(z,t), \quad E = E(x,y,z,t) \end{aligned}\right\} \qquad \ldots(1.1)$$

which reduces to the space-time of Peres (1959) when $A = B = 1$ and $E = E(x, y, Z)$, $(Z = z - t)$. The space-time defined by (1.1) and some of its important properties have been discussed in a paper by Pandey (1975). The purpose of the present paper is to obtain some plane wave-like solutions in the sense of Takeno (1961) of the field equations in non-symmetric unified field theories of Einstein, Bonnor and Schrödinger corresponding to the solutions given by Lal and Pandey (1974) by using the methods of Takeno (1961 *a*, 1956, 1957) and Prasad and Lal (1965).

The field equations of Einstein's unified theory (Einstein 1951) are

$$g_{ij,k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{kj} = 0, \qquad (\,,k = \partial/\partial x^k), \qquad \ldots(1.2)$$

$$\Gamma^k_{[ik]} = 0, \qquad (E_1) \qquad \ldots(1.3)$$

$$(a)\; R_{(ik)} = 0, \quad (b)\; R_{[ij],k} + R_{[jk],i} + R_{[ki],j} = 0, \qquad \ldots(1.4)$$

* Presented by the author at the 5th Indian G.R.G. Conference held at I.I.T., Bombay, December 27–29, 1974.

** Present address: Hulopino–13, Korpus–1, Leningrad K–21, U.S.S.R.

$$\left.\begin{array}{l}(1.2),\ (1.3),\\ R_{ik}=0,\end{array}\right\}\ (E_2) \qquad \ldots(1.5)$$

where

$$R_{ik}=\Gamma^{s}_{ik,s}-\Gamma^{s}_{is,k}-\Gamma^{s}_{tk}\,\Gamma^{t}_{is}+\Gamma^{s}_{ts}\,\Gamma^{t}_{ik}, \qquad \ldots(1.6)$$

is the generalized Ricci tensor and a small bracket and a square bracket enclosing a pair of indices denote the symmetry and skew-symmetry respectively between them. (E_1) and (E_2) are known as weak and strong field equations.

The field equations of Bonnor (1954) are given by (E_1) when R_{ik} is replaced by $R_{ik}+p^2U_{ik}$, where p is an arbitrary real or imaginary constant and

$$U_{ik}=g_{[ki]}-g^{[mn]}\,g_{im}\,g_{nk}+(\tfrac{1}{2})\,g^{[mn]}\,g_{nm}\,g_{ik}; \qquad \ldots(1.7)$$

and that of Schrödinger's non-symmetric unified field theory (Schrödinger 1950) are given by (E_1) and (E_2) when R_{ik} is replaced by $R_{ik}+\lambda g_{ik}$, where λ is a non-vanishing constant.

These theories are based on geometrical interpretation of gravitation and electromagnetism by using a four dimensional generalized Riemannian space-time of which the non-symmetric fundamental tensor g_{ik} consists of symmetric part coinciding with the fundamental tensor of space-time representing the gravitational potential and the skew-symmetric part signifying the electromagnetic field. Therefore, following Takeno (1956, 1957, 1961*a*) and Prasad and Lal (1965), we assume:

(I) The line element of the Riemannian space-time (*i.e.*, the space-time defined by $g_{(ik)}=h_{ik}$) is given by (1.1), where A, B, E are to be determined later.

(II) The anti-symmetric part of g_{ik} (*i.e.*, $g_{[ik]}=f_{ik}$) is given by

$$f_{12}=f_{34}=0,\quad f_{13}=-f_{14}=\rho,\quad f_{23}=-f_{24}=\sigma, \qquad \ldots(1.8)$$

where ρ and σ are functions of x, y and Z to be determined later.

Therefore, from (1.1) and (1.8) we have

$$(a)\quad g_{ik}=\begin{pmatrix}-A & 0 & \rho & -\rho\\ 0 & -B & \sigma & -\sigma\\ -\rho & -\sigma & -(1-E) & -E\\ \rho & \sigma & -E & 1+E\end{pmatrix},$$

$$(b)\quad g^{ik}=\begin{pmatrix}-1/A & 0 & \rho/A & \rho/A\\ 0 & -1/B & \sigma/B & \sigma/B\\ -\rho/A & -\sigma/B & -1-E+F & -E+F\\ -\rho/A & -\sigma/B & -E+F & 1-E+F\end{pmatrix}, \qquad \ldots(1.9)$$

where $F=\rho^2/A+\sigma^2/B$. Also $g=h=-AB$, $[h\equiv\det.(h_{ik})]$.

2. Solutions of (1.2) and (1.3)

Let us put

$$\Gamma^k_{ij} = p^k_{ij} + q^k_{ij}, \quad p^k_{ij} = \Gamma^k_{(ij)}, \quad q^k_{ij} = \Gamma^k_{[ij]} \qquad \text{...(2.1)}$$

and define p's in terms of q's after Takeno *et al.* (1951) by

$$p^k_{ij} = \begin{Bmatrix} k \\ ij \end{Bmatrix} + h^{kl}\left(q^m_{li} f_{jm} + q^m_{lj} f_{im}\right) \qquad \text{...(2.2)}$$

where $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ is calculated from h_{ij} and h^{ij} is defined by $h_{ij}\, h^{ik} = \delta_j{}^k$. Inserting the relevant quantities from (1.9) into (2.2) we find p's in terms of q's as:

$$p^1_{11} = 0, \quad p^2_{11} = -(2\rho/B)(q^3_{21} - q^4_{21}), \quad {}^{3}_{11} = -(1+E)A,_3/2$$
$$- EA,_4/2 - 2\rho(1+E)(q^3_{31} - q^4_{31}) - 2E\rho(q^3_{41} - q^4_{41}), \quad p^4_{11} = -EA,_3/2$$
$$+ (1-E)A,_4/2 - 2E\rho(q^3_{31} - q^4_{31}) + 2(1-E)\rho(q^3_{41} - q^4_{41})$$
$$p^1_{12} = -(\rho/A)(q^3_{12} - q^4_{12}), \quad p^2_{12} = -(\sigma/B)(q^3_{21} - q^4_{21}),$$
$$p^3_{12} = -(1+E)\{\sigma(q^3_{31} - q^4_{31})$$
$$+ \rho(q^3_{32} - q^4_{32})\} - E\{\sigma(q^3_{41} - q^4_{41}) + \rho(q^3_{42} - q^4_{42})\}, \qquad \text{...(2.3)}$$

and similar expressions for p^k_{13}, p^k_{14}, etc., where $k = 1, \ldots, 4$, are omitted for brevity's sake.

Using (2.1) and (2.3) in (1.2), the components of q^k_{ij} are given by

$$q^3_{34} = q^4_{34} = 0, \quad q^1_{12} = \sigma\theta A/2A, \quad q^2_{12} = -\rho\theta B/2B,$$
$$q^3_{12} = q^4_{12} = -(\rho,_2 - \sigma,_1)/2,$$
$$q^1_{13} = -q^1_{14} = -\rho,_1/A, \quad q^2_{13} = -q^2_{14} = -(\rho,_2 + \sigma,_1)/2B,$$
$$q^2_{23} = -q^2_{24} = -\sigma,_2/B$$
$$q^1_{23} = -q^1_{24} = -(\rho,_2 + \sigma,_1)/2A, \; q^3_{13} = -q^3_{14} = M, \; q^1_{34} = -\rho\theta A/2A^2,$$
$$q^2_{34} = -\sigma\theta B/2B^2,$$
$$q^3_{23} = -q^3_{24} = N, \quad q^4_{13} = -q^4_{14} = M - \rho\theta A/2A, \quad q^4_{23} = q^4_{24} = N$$
$$-\sigma\theta B/2B \qquad \text{...(2.4)}$$

where

$$\theta = \partial/\partial z + \partial/\partial t, \quad A,_3 = \partial A/\partial z, \quad w = \theta E, \quad \rho,_1 = \partial\rho/\partial x, \text{ etc.},$$
$$M = -\rho,_3 + \rho A,_3/2A + \rho w/2 + (\rho\theta A/2A^2)(EA + \rho^2) + \rho\sigma^2\theta B/2B^2$$
$$- \rho\sigma^2(\theta A/A - \theta B/B)/2B$$
$$N = -\sigma,_3 + \sigma B,_3/2B + \sigma w/2 + (\sigma\theta B/2B^2)(EB + \sigma^2) + \sigma\rho^2\theta A/2A^2$$
$$+ \sigma\rho^2(\theta A/A - \theta B/B)/2A.$$

Using (2.4) (2.3) in the components of p^k_{ij} are obtained in terms of A, B, E, ρ, σ and their derivatives. The components of Γ^k_{ij} can now easily be calculated from (2.1). This completes the solution of (1.2).

It is seen that Γ^k_{ij} satisfies

$$\left.\begin{aligned} &\Gamma^s_{1s} = -\Gamma^s_{s1} = \rho\,(\partial A/A - \partial B/B)/2,\ \Gamma^s_{2s} = -\Gamma^s_{s2} \\ &\quad = -\sigma\,(\partial A/A - \partial B/B)/2 \\ &\Gamma^s_{3s} - v = \Gamma^s_{s3} + v = (A,_3/A + B,_3/B)/2 \\ &\Gamma^s_{4s} + v = \Gamma^s_{s4} - v = (A,_4/A + B,_4/B)/2,\ (v = \rho,_1/A + \sigma,_2/B). \end{aligned}\right\} \quad \ldots(2.5)$$

In view of (2.5) the equation (1.3) is equivalent to

$$(a)\ \partial A/A = \partial B/B, \qquad (b)\ v = 0. \qquad \ldots(2.6)$$

In order to find the solutions of the remaining field equations, we first determine A and B from (2.6*a*). Two cases are immediately suggested:

Case I: $\partial A/A = \partial B/B = 0$; giving A, B to depend on z and t through their composite form $Z \equiv z - t$;

Case II: $\partial A/A = \partial B/B \neq 0$; giving $A = B \exp(\phi)$, ϕ being some function of Z.

In this paper we have obtained the solutions for Case I only as they correspond to plane wave-like solutions in the sense of Takeno (1961). The solutions for Case II can be found similarly but the calculations will be too much. The solutions under Case II may not be plane wave-like in the sense of Takeno.

3. Solutions of (E_1) and (E_2)

The non-vanishing components of R_{ik} for Case I are given by

$$\left.\begin{aligned} &-R_{13} = R_{14} = (\Delta\rho + w,_1)/2, \quad -R_{23} = R_{24} = (\Delta\sigma + w,_2)/2 \\ &-R_{31} = R_{41} = (-\Delta\rho + w,_1)/2, \quad -R_{32} = R_{42} = (-\Delta\sigma + w,_2)/2 \\ &R_{34} = R_{43} = -R_{33} - \partial w/2 = -R_{44} + \partial w/2 = W + (\rho/A)\,\Delta\rho \\ &\quad + (\sigma/B)\,\Delta\sigma - \partial wE/2 \end{aligned}\right\} \quad \ldots(3.1)$$

where

$$\begin{aligned} \Delta &= (1/A)\,\partial^2/\partial x\partial x + (1/B)\,\partial^2/\partial y\partial y, \quad W = -\Delta E/2 \\ &- w\,(A'/A + B'/B)/4 + (1/2AB)\,(\sigma,_1 + \rho,_2)^2 + (\rho,_1/A)^2 \\ &+ (\sigma,_2/B)^2 + (A''/A + B''/B)/2 - \big((A'/A)^2 + (B'/B)^2\big)/4, \\ &(A' = \partial A/\partial Z, \quad \text{etc.}). \end{aligned}$$

It should be seen that under the new time coordinate $Z \equiv z - t$ (*cf.* Case I) $g_{(ik)} = h_{ik}$ transforms to the form

$$\left.\begin{array}{l} ds^2 = -Adx^2 - Bdy^2 - 2dzdZ + (1+E)\,dZ^2 \\ A = A(Z), \quad B = B(Z), \quad E = E(x, y, z, Z) \end{array}\right\} \qquad \ldots(3.2)$$

which under the choice

$$A = B = A(Z) \qquad \ldots(3.3)$$

reduces to the form (of course with different orientation of dx^i, $x^i \equiv x, y, z, t$) discussed by Vaidya (1961) in which he has investigated the various plane wave solutions of (E_1) and (E_2). Thus (1.9*a*) under the new time coordinate becomes

$$g_{ik} = \begin{pmatrix} -A & 0 & \rho & -\rho \\ 0 & -B & \sigma & -\sigma \\ -\rho & -\sigma & 0 & -1 \\ \rho & \sigma & -1 & 1+E \end{pmatrix}, \qquad \ldots(3.4)$$

where $A = A(Z)$, $B = B(Z)$, $E = E(x, y, z, Z)$, $\rho = \rho(x, y, Z)$, $\sigma = \sigma(x, y, Z)$.

If we assume in (3.4)

$$A = B = 1, \quad E = 0, \quad \rho = \rho(Z), \quad \sigma = \sigma(Z), \qquad \ldots(3.5)$$

and

$$A, B, \rho \text{ and } \sigma \text{ as in (3.5) and } E = E(x, y, Z) \qquad \ldots(3.6)$$

such that E is harmonic function of x and y, we get respectively the solutions I and II of Vaidya (1961) [as $R_{ik} = 0$ and both (E_1) and (E_2) are satisfied] corresponding to propagation of plane electromagnetic waves and unified waves. We shall, now, give a more general solution of (E_1) and (E_2) in the coordinate system of (1.9).

The components of $R_{(ik)}$ [obtainable from (3.1)] when substituted in equation (1.4*a*) give

$$(a)\; w,_1 = 0, \qquad (b)\; w,_2 = 0, \quad (c)\; \partial w = 0, \qquad \ldots(3.7)$$

$$W + (\rho/A)\,\Delta\rho + (\sigma/B)\,\Delta\sigma = 0. \qquad \ldots(3.8)$$

Integrating (3.7) we get E in the form

$$E = H(x, y, Z) + zf(Z) \qquad \ldots(3.9)$$

where H and f are functions of (x, y, Z) and Z respectively. Similarly the components of $R_{[ik]}$ when substituted in (1.4*b*) yield

$$\Delta(\rho,_2 - \sigma,_1) = 0. \qquad \ldots(3.10)$$

It may be noticed that ρ and σ can be determined from (2.6*b*) and (3.10). Hence we have:

The g_{ik} given by (1.9) where A and B are functions of Z and E given by (3.9) is a solution of (E_1) if A, B, E, ρ and σ satisfy (2.6*b*), (3.8) and (3.10).

In view of (3.1) the equation (1.5) yields (3.7) and

$$\Delta\rho = 0, \quad \Delta\sigma = 0, \quad W = 0. \qquad ...(3.11)$$

Hence we have:

The g_{ik} given by (1.9) where A and B are functions of Z and E given by (3.9) is a solution of (E_2) if A, B, E, ρ and σ satisfy (2.6b) and (3.11).

4. Solutions of Bonnor's and Schrödinger's Field Equations

The components of U_{ik} obtained from (1.7) and (1.9) satisfy

$$U_{33} = -U_{34} = -U_{43} = U_{44} = -2F, \quad \text{other} \quad U_{(ik)} = 0 \qquad ...(4.1a)$$

$$\left.\begin{aligned} U_{[13]} &= -U_{[31]} = -U_{[14]} = U_{[41]} = -2\rho \\ U_{[23]} &= -U_{[32]} = -U_{[24]} = U_{[42]} = -2\sigma, \quad \text{other} \quad U_{[ik]} = 0 \end{aligned}\right\} \qquad ...(4.1b)$$

which together with $R_{(ik)}$ and $R_{[ik]}$ when used in the field equations of Bonnor yield (3.7) [accordingly E has the form given by (3.9)] and

$$W + (\rho/A)\,\Delta\rho + (\sigma/B)\,\Delta\sigma + 2p^2 F = 0 \qquad ...(4.2)$$

$$\Delta' (\rho,_2 - \sigma,_1) = 0, \; \left(\Delta' = (1/2)\,\Delta + 2p^2\right). \qquad ...(4.3)$$

Hence we have:

The g_{ik} given by (1.9) where A and B are functions of Z and E given by (3.9) is a solution of Bonnor's unified field theory if A, B, E, ρ and σ satisfy (2.6b), (4.2) and (4.3).

It can easily be seen that (1.9), in view of (3.1), is not the solution of Schrödinger's non-symmetric unified field theory. It is, in fact, due to λ being a non-vanishing constant.

Here it should be mentioned that both [(3.4) with (3.5)] and [(3.4) with (3.6)] satisfy (4.3) identically and reduce (4.2) to $p^2 F = 0$ which implies that either $p = 0$ or $F = 0$ or both p and F vanish. When p vanishes, the field equations of Bonnor reduce to (E_1). But $F \equiv \rho^2 + \sigma^2 = 0$ (as $A = B = 1$) implies that $\rho = \sigma = 0$. Thus when both p and F vanish we get pure gravitational fields because (E_1) when non-symmetric field vanishes, reduces to Einstein's vacuum field equation $K_{ij} = 0$.

5. A Special Case

If we consider ρ and σ satisfying the Maxwell's field equations in general relativity, then as shown by Lal and Pandey (1974), they satisfy in the coordinate system of (1.1)

$$(a)\; \rho,_1 + \sigma,_2 = 0, \quad (b)\; A\,\rho,_2 - B\,\sigma,_1 = 0, \qquad ...(5.1)$$

and accordingly from (5.1) and [(5.1a) and (2.6b)] we get respectively

$$(a)\; \Delta\rho = \Delta\sigma = 0, \quad \text{and} \quad (b)\; A = B, \; (\text{as } \rho,_1 \neq 0, \;\; \sigma,_2 \neq 0). \qquad ...(5.2)$$

Thus it can be seen that ρ and σ, satisfying Cauchy-Riemann type equations [(5.1) with (5.2 *b*)], satisfy both (E_1) and (E_2) when $W = 0$. Hence we have:

A necessary and sufficient condition that g_{ik} given by (1.9) where $A\ (= B)$ is a function of Z and E given by (3.9) be a solution of Einstein's unified field theory is that A, E, ρ and σ satisfy

$$\nabla H = 2A'' - A'(f + A'/A) + 4(\rho^2{}_{,1} + \rho^2{}_{,2})/A, \quad (= \nabla \partial^2/\partial x^2 + \partial^2/\partial y^2). \qquad \ldots(5.3)$$

Note that here $A = B$ is a consequence of ρ and σ satisfying Maxwell's field equations; and this result follows immediately. Similarly for this choice of ρ and σ one can easily obtain the following result for Bonnor's unified field theory.

A necessary and sufficient condition that g_{ik} given by (1.9) where $A\ (= B)$ is a function of Z and E given by (3.9) be a solution of Bonnor's unified field theory is that A, E, ρ and σ satisfy

$$\nabla H + 2p^2 F^* = 2A'' - A'(f + A'/A) + 4(\rho^2{}_{,1} + \rho^2{}_{,2})/A,$$
$$\left(F^* = (\rho^2 + \sigma^2)/A\right). \qquad \ldots(5.4)$$

6. Concluding Remarks

We shall compare the solution in general relativity (GR) given by Lal and Pandey (1974) with the solution of non-symmetric theory (NS) obtained in section 3. The solution in GR is (1.1) where A, B, and E are as obtained here, *i.e.*, A, B depend on Z and E is a function of x, y, z and t of the form given by (3.9); and they satisfy (5.1) and

$$-W + (\sigma_{,1} + \rho_{,2})^2/2AB + (\rho_{,1}/A)^2 + (\sigma_{,2}/B)^2 = 8\pi(\rho^2/B + \sigma^2/A). \qquad \ldots(6.1)$$

In both solutions A, B, E are related to F_{ik} by some equations. That is, they must satisfy (5.1) and (6.1) in the former, while in the solutions of (E_1) and (E_2) they are connected with F_{ik} by [(3.8), (3.10)] and (3.11) respectively. Therefore, in general the (A, B, E)'s in the solution of GR cannot be the same as (A, B, E)'s in the solutions of NS even though we take the same ρ and σ.

Lastly, we add that in case we take the function $f = 0$, E will reduce to Peres function [*cf.* (3.9)]. Thus by setting $A = B = 1$ and taking for E (3.9) with $f = 0$, the solutions of sections 3 and 4 reduce to the solutions of the corresponding non-symmetric unified field theories in Peres space-time.

Acknowledgement

The author sincerely thanks Professor P. C. Vaidya for his suggestions which ed to its improvement.

REFERENCES

Bonnor, W. B. (1954). The equation of motion of the non-symmetric unified field theory. *Proc. R. Soc.*, A **226**, 366.

Einstein, A. (1951). The Meaning of Relativity, 5th ed. London, App. II.

Lal, K. B., and Pandey, S. N. (1974). Wave solution of the field equations of general relativity in a generalized Peres space-time. *Tensor, N.S.*, **28**, 164.

Pandey, S. N. (1975). On a generalized Peres space-time. *Indian J. pure appl. Math.* (to appear).

Patel, L. K., and Vaidya, P. C. (1971). A generalized plane wave metric. *Ann. Inst. Henri Poincaré, XV*, no. 2, 169.

Peres, A. (1959). Some gravitational waves. *Phys. Rev. Lett.*, **3**, 571.

Prasad, H., and Lal, K. B. (1965). On cylindrical wave solutions of Einstein's unified field theory of gravitation and electromagnetism. *Tensor, N.S.*, **16**, 209.

Schrödinger, E. (1950). Space-time Structure. Cambridge.

Takeno, H. (1956). Some wave solutions of Einstein's generalized theory of gravitation. *Tensor, N.S.*, **6**, 69.

——— (1957). On some generalized plane wave solutions of non-symmetric unified field theory. *Tensor, N.S.*, **7**, 34.

——— (1958). A comparison of plane wave solutions in general relativity with those in non-symmetric theory. *Prog. theor. Phys.*, **20**, 267.

——— (1961). The mathematical theory of plane gravitational waves in general relativity. *Sci. Rep. Res. Inst. Theor. Phys., Hiroshima University*, **1**.

——— (1961*a*). Some plane wave-like solutions of non-symmetric unified field theory. *Tensor, N.S.*, **11**, 263.

Takeno, H., Ikeda, M., and Abe, S. (1951). On solutions of new field equations of Einstein and those of Schrödinger. *Prog. theor. Phys.*, **6**, 837.

Vaidya, P. C. (1961). Unified gravitational and electromagnetic waves. *Prog. theor. Phys.*, **25**, 305.

Vaidya, P. C., and Pandya, I. M. (1960). Radiation: gravitational and electromagnetic. *Curr. Sci*, **29**, 268.