MIXED BOUNDARY VALUE PROBLEM FOR AN ELASTIC WEDGE

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Integral transforms have been used to reduce a mixed boundary value problem in an infinite wedge-shaped medium when one plane face only is separated sectorwise by mixed conditions, to a Fredholm integral equation of second kind. For approximate solution, Wiener-Hopf technique has been employed.

1. INTRODUCTION

Most of the diffraction problems in wave propagation and statical problems of elasticity under mixed boundary conditions have been dealt with by Wiener-Hopf technique. Details of the use of the technique can be had in Noble (1958). Boundary value problems associated with the media like wedge, cone or their parts involve Fourier, Mellin, Lebedev-Kontorovich or Mehler-Fok transforms [*cf.* Jones (1950), Matczynski (1962), Pridmore Brown (1968), Nuller (1970), Aleksandrov and Chebakov (1972), Low (1966) and others]. Legendre transform is also useful in solving certain classes of problems under mixed boundary conditions, particularly, when the mixed type of conditions are given for different ranges in spherical polar, spheroidal or toroidal coordinates. The corresponding Wiener-Hopf equation of this type of problem can conveniently be solved if one uses Legendre transformation formula where the degree of the associated Legendre function is an integration variable [*cf.* Felsen (1958)] or a generalized Mehler-Fok transform. Applications of the transformation formula involving degree of Legendre function can be had in Mandal (1975).

The success of solving the said Wiener-Hopf equation depends on the factor ization of its coefficients. The author notices that mixed boundary value problems in wedge-shaped medium usually give rise to complex form of coefficients in the corresponding Wiener-Hopf equations. An interesting approximate method for decomposition of the coefficients has been suggested by Koiter (1954) and used by Matczynski (1962) and others.

In the present paper a mixed boundary value problem is considered in an infinite wedge-shaped medium when the mixed condition divides one plane face into two sectors. The problem is then reduced to the solution of a Fredholm integral equation of second kind by successive application of Mellin and Legendre transforms. In the last section of the paper, the integral equation obtained is solved by Wiener-Hopf technique by approximating one of its coefficients by the method suggested by Koiter (1954).

2. Formulation of the Problem

We consider an infinite wedge-shaped elastic medium bounded by the planes $\phi = 0$ and $\phi = \gamma$, γ being the wedge angle given by $0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant \gamma$. The boundary conditions are

$$u_r = u_\theta = u_\phi = 0, \quad \text{on } \phi = 0 \qquad \ldots(2.1)$$

$$u_r = u_\theta = 0, \quad \text{on } \phi = \gamma \qquad \ldots(2.2)$$

$$u_\phi = f_1(r, \theta), \quad \text{on } \phi = \gamma, \quad 0 \leqslant \theta < \alpha \qquad \ldots(2.3)$$

$$\sigma_{\phi\phi} = 0, \quad \text{on } \phi = \gamma, \quad \alpha < \theta \leqslant \pi. \qquad \ldots(2.4)$$

3. Solution of Equilibrium Equation and Reduction of the Boundary Conditions

In spherical polar coordinates the Papkovich-Neuber form of solutions [*cf.* Lur'e (1964)] of the equilibrium equation of the theory of elasticity in displacements are

$$2\mu_1 u_r = -\frac{\partial F}{\partial r} + 4(1-\nu_1)[\sin\theta\cos\phi \,.\, \phi_1 + \sin\theta\sin\phi \,.\, \phi_2 + \cos\theta \,.\, \phi_3] \qquad \ldots(3.1)$$

$$2\mu_1 u_\theta = -\frac{1}{r}\frac{\partial F}{\partial \theta} + 4(1-\nu_1)[\cos\theta\cos\phi \,.\, \phi_1 + \cos\theta\sin\phi \,.\, \phi_2 - \sin\theta \,.\, \phi_3] \qquad \ldots(3.2)$$

and

$$2\mu_1 u_\phi = -\frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi} + 4(1-\nu_1)[-\sin\phi \,.\, \phi_1 + \cos\phi \,.\, \phi_2], \qquad \ldots(3.3)$$

where

$$F = r\sin\theta\cos\phi \,.\, \phi_1 + r\sin\theta\sin\phi \,.\, \phi_2 + r\cos\theta \,.\, \phi_3 + \phi_0 \qquad \ldots(3.4)$$

and ϕ_i, $i = 0, 1, 2, 3$ are harmonic potentials and μ_1, ν_1 are the rigidity modulus and Poisson's ratio respectively of the material of the elastic region under consideration. The normal stress $\sigma_{\phi\phi}$ in terms of the above potential functions is given by

$$\begin{aligned}
\sigma_{\phi\phi} = {} & 2(1-\nu_1)\left[\frac{1}{r\sin\theta}\left(-\sin\phi \,.\, \frac{\partial\phi_1}{\partial\phi} + \cos\phi \,.\, \frac{\partial\phi_2}{\partial\phi}\right)\right] \\
& + 2\nu_1\left[\left(\sin\theta\cos\phi\frac{\partial\phi_1}{\partial r} + \sin\theta\sin\phi \,.\, \frac{\partial\phi_2}{\partial r} + \cos\theta\frac{\partial\phi_3}{\partial r}\right)\right. \\
& \left. + \frac{1}{r}\left(\cos\theta\cos\phi\frac{\partial\phi_1}{\partial\theta} + \cos\theta\sin\phi\frac{\partial\phi_2}{\partial\theta} - \sin\theta\frac{\partial\phi_3}{\partial\theta}\right)\right] \\
& - \left[\sin\theta\cos\phi\left(\frac{1}{r\sin^2\theta}\frac{\partial^2\phi_1}{\partial\phi^2} + \frac{\partial\phi_1}{\partial r} + \frac{\cot\theta}{r}\frac{\partial\phi_1}{\partial\theta}\right) + \right.
\end{aligned}$$

(*equation continued on p. 141*)

$$+ \sin\theta \sin\phi \left(\frac{1}{r \sin^2\theta} \frac{\partial^2 \phi_2}{\partial \phi^2} + \frac{\partial \phi_2}{\partial r} + \frac{\cot\theta}{r} \frac{\partial \phi_2}{\partial \theta} \right)$$

$$+ \cos\theta \left(\frac{1}{r \sin^2\theta} \frac{\partial^2 \phi_3}{\partial \phi^2} + \frac{\partial \phi_3}{\partial r} + \frac{\cot\theta}{r} \frac{\partial \phi_3}{\partial \theta} \right)$$

$$+ \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi_0}{\partial \phi^2} + \frac{1}{r} \frac{\partial \phi_0}{\partial r} + \frac{\cot\theta}{r^2} \frac{\partial \phi_0}{\partial \theta} \Big]. \quad \ldots(3.5)$$

The other stress components, *e.g.*, σ_{rr}, $\sigma_{\theta\theta}$, $\tau_{r\theta}$, $\tau_{r\phi}$ and $\tau_{\theta\phi}$ can be calculated in terms of the above potential functions by using the stress strain relations. The four potential functions ϕ_i, $i = 0, 1, 2, 3$ are not all independent. One of them can be chosen arbitrarily. For the sake of definiteness let us choose

$$\phi_3 = 0, \quad \text{on} \quad \phi = 0. \quad i.e., \quad \phi_{30} = 0. \quad \ldots(3.6)$$

This fixes only an initial (boundary) value of ϕ_3. One of the four ϕ_i's can still be chosen arbitrarily subject to (3.6).

Then the first two boundary conditions in (2.1) under (3.1) and (3.2) give

$$\left[-\frac{\partial F}{\partial r} \right]_{\phi=0} + 4(1-\nu_1)\sin\theta \,.\, \phi_{10} = 0 \quad \ldots(3.7)$$

and

$$\left[-\frac{\partial F}{\partial \theta} \right]_{\phi=0} + 4(1-\nu_1)\, r\cos\theta \,.\, \phi_{10} = 0.$$

Eliminating F between these equations we get

$$\sin\theta \frac{\partial}{\partial \theta} \phi_{10} - r\cos\theta \frac{\partial}{\partial r} \phi_{10} = 0. \quad \ldots(3.8)$$

From (3.6),

$$\left[\frac{\partial \phi_3}{\partial r} \right]_{\phi=0} = \frac{\partial \phi_{30}}{\partial r} = 0. \quad \ldots(3.9)$$

Therefore, from (3.7) and (3.9) we get,

$$r \frac{\partial \phi_{00}}{\partial r} = \sin\theta \left[-r \frac{\partial \phi_{10}}{\partial r} + (3-4\nu_1)\phi_{10} \right], \quad \ldots(3.10)$$

where the posterior suffix zero indicates the value of potential on $\phi = 0$. Again the last boundary condition in (2.1) under (3.3) gives,

$$(3-4\nu_1)\phi_{20} = \left[\frac{\partial \phi_1}{\partial \phi} + \cot\theta \frac{\partial \phi_3}{\partial \phi} + \frac{1}{r\sin\theta} \frac{\partial \phi_0}{\partial \phi} \right]_{\phi=0}. \quad \ldots(3.11)$$

In the above derivations, and in the sequel, we denote the value of ϕ_i on $\phi = 0$ by ϕ_{i0} and the value of ϕ_i on $\phi = \gamma$ by $\phi_{i\gamma}$, $i = 0, 1, 2, 3$.

We choose so that

$$F = 0, \quad \text{for } \phi = \gamma. \quad \ldots(3.12)$$

Then

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial \theta} = 0, \quad \text{for } \phi = \gamma. \qquad \ldots(3.13)$$

So the boundary conditions (2.2) under (3.1) and (3.2) are reduced to

$$\cos\gamma \,.\, \phi_{1\gamma} + \sin\gamma \,.\, \phi_{2\gamma} = 0 \qquad \ldots(3.14)$$

and

$$\phi_{3\gamma} = 0. \qquad \ldots(3.15)$$

Again from (3.12), (3.14) and (3.15) we get

$$\phi_{0\gamma} = 0. \qquad \ldots(3.16)$$

The boundary condition (2.3) under (3.12) and (3.3) becomes

$$2\mu_1 f_1(r, \theta) = -\frac{1}{r\sin\theta}\left[\frac{\partial F}{\partial \phi}\right]_{\phi=\gamma} + 4(1-\nu_1)\left[-\sin\gamma \,.\, \phi_{1\gamma} + \cos\gamma \,.\, \phi_{2\gamma}\right],$$

$$\text{for } 0 \leqslant \theta < \alpha. \qquad \ldots(3.17)$$

Also the boundary condition (2.4) under (3.12) and (3.5) after some simplification is

$$-\sin\gamma\left[\frac{\partial \phi_1}{\partial \phi}\right]_{\phi=\gamma} + \cos\gamma\left[\frac{\partial \phi_2}{\partial \phi}\right]_{\phi=\gamma} = 0, \quad \text{for} \quad \alpha < \theta \leqslant \pi. \qquad \ldots(3.18)$$

4. Method of Solution

Reduction of the Boundary Value Problem to a Fredholm Integral Equation of Second Kind

The Mellin transform $\bar{f}(p)$ of the function $f(r)$ is defined by

$$\bar{f}(p) = \int_0^\infty f(r)\, r^{p-1}\, dr, \qquad \ldots(4.1)$$

and its inversion formula is given by

$$f(r) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{f}(p)\, r^{-p}\, dp, \qquad \ldots(4.2)$$

where $c > \operatorname{Re} p > 0$ and is such that $\int_0^\infty r^{p-1}\,|f(r)|\,dr$ is bounded.

Applying this transform to the boundary condition (3.6), it becomes

$$\bar{\phi}_{30}(p) = 0. \qquad \ldots(4.3)$$

Again (3.7), after taking its Mellin transform, becomes

$$\sin\theta \frac{\partial \bar{\phi}_{10}}{\partial \theta} - \cos\theta \int_0^\infty r^p \frac{\partial \phi_{10}}{\partial r}\, dr = 0,$$

or,

$$\sin\theta \frac{\partial \bar{\phi}_{10}}{\partial \theta} + p\cos\theta\, \bar{\phi}_{10} = \cos\theta\, [r^p \phi_{10}]_0^\infty,$$

or,

$$\sin\theta \frac{\partial \bar{\phi}_{10}}{\partial \theta} + p\cos\theta\, \bar{\phi}_{10} = 0, \qquad \text{...(4.4)}$$

on the assumption here and in the sequel that terms like

$$[r^p \phi_{10}]$$

vanish at both the limits 0 and ∞. This requirement is satisfied if one takes p in the form

$$p = \tfrac{1}{2} + \nu, \qquad \text{...(4.5)}$$

ν having a very small real part.

The boundary condition (3.10) is

$$-p\,\bar{\phi}_{00} = p\sin\theta\,\bar{\phi}_{10} + (3 - 4\nu_1)\,\bar{\phi}_{10}$$

or,

$$\bar{\phi}_{00} = -\frac{1}{p}\,[p\sin\theta + 3 - 4\nu_1]\,\bar{\phi}_{10}. \qquad \text{...(4.6)}$$

Also, from (3.11) one obtains

$$(3 - 4\nu_1)\,\bar{\phi}_{20} = \left[\frac{\partial}{\partial\phi}\left(\bar{\phi}_1 + \cot\theta\,.\,\bar{\phi}_3 + \frac{p-1}{\sin\theta}\,\bar{\phi}_0\right)\right]_{\phi=0}. \qquad \text{...(4.7)}$$

Similarly, after taking their Mellin transforms the boundary conditions (3.14) to (3.18) on $\phi = \gamma$ respectively become

$$\cos\gamma\,.\,\bar{\phi}_{1\gamma} + \sin\gamma\,.\,\bar{\phi}_{2\gamma} = 0, \qquad \text{...(4.8)}$$

$$\bar{\phi}_{3\gamma} = 0, \qquad \text{...(4.9)}$$

$$\bar{\phi}_{0\gamma} = 0, \qquad \text{...(4.10)}$$

$$\left[-\left(\cos\gamma\,.\,\frac{\partial\bar{\phi}_1}{\partial\phi} + \sin\gamma\,.\,\frac{\partial\bar{\phi}_2}{\partial\phi}\right) - \cot\theta\,\frac{\partial\bar{\phi}_3}{\partial\phi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\bar{\phi}_0\,(p-1)\right]_{\phi=\gamma}$$

$$+ (3 - 4\nu_1)\,[-\sin\gamma\,.\,\bar{\phi}_{1\gamma} + \cos\gamma\,.\,\bar{\phi}_{2\gamma}] = \bar{f}(p,\theta), \qquad \text{...(4.11)}$$

for $0 \leqslant \theta < \alpha$, where $\bar{f}(p, \theta)$ is the Mellin transform of $2\mu_1 f_1(r, \theta)$; and

$$\left[-\sin\gamma \cdot \frac{\partial \bar{\phi}_1}{\partial \phi} + \cos\gamma \frac{\partial \bar{\phi}_2}{\partial \phi}\right]_{\phi=\gamma} = 0, \quad \text{for } \alpha < \theta \leqslant \pi. \qquad ...(4.12)$$

Since ϕ_i's $(i = 0, 1, 2, 3)$ are harmonic functions, they are to satisfy the equation

$$\left[r^2 \frac{\partial^2}{\partial r^2} + 2r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2}\right] \phi_i = 0.$$

Taking its Mellin transform one obtains,

$$\left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + p(p-1)\right] \bar{\phi}_i = 0,$$

or,

$$\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial \bar{\phi}_i}{\partial \theta}\right] + \frac{1}{\sin^2\theta} \frac{\partial^2 \bar{\phi}_i}{\partial \phi^2} + \left(\nu^2 - \frac{1}{4}\right) \bar{\phi}_i = 0,$$

$$(i = 0, 1, 2, 3) \qquad ...(4.13)$$

by (4.5).

Solution of ϕ_3

By virtue of (4.3) and (4.9), the equation (4.13) gives immediately

$$\bar{\phi}_3 \equiv 0,$$

and therefore, from (4.2) one obtains

$$\phi_3(r, \theta, \phi) \equiv 0. \qquad ...(4.14)$$

Solution of ϕ_0

Since the boundary condition (4.4) is true for all values of θ on $\phi = 0$ and since $\bar{\phi}_{10}$ is finite, it gives

$$\bar{\phi}_{10} = 0. \qquad ...(4.15)$$

Employing (4.15) in (4.6) one obtains

$$\bar{\phi}_{00} = 0. \qquad ...(4.16)$$

Now from (4.10) and (4.16) one gets

$$\bar{\phi}_0 \equiv 0,$$

and hence by (4.2) it gives

$$\phi_0(r, \theta, \phi) \equiv 0. \qquad ...(4.17)$$

Solution of ϕ_1 and ϕ_2

The harmonic potentials $\bar{\phi}_1$, $\bar{\phi}_2$ satisfying the differential equation (4.13) can be determined by the boundary conditions (4.15), (4.7), (4.8), (4.11) and (4.12).

Let us represent $\bar{\phi}_i$, $i = 1, 2$ in the following manner:

$$\bar{\phi}_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tfrac{1}{2} + \nu + \mu)\, \Gamma(\tfrac{1}{2} - \nu + \mu)\, [A(\mu) \cos \mu\phi + B(\mu) \sin \mu\phi] \times P_{\nu-\frac{1}{2}}^{-\mu}(\cos \theta)\, \mu d\mu, \quad ...(4.18)$$

and

$$\bar{\phi}_2 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tfrac{1}{2} + \nu + \mu)\, \Gamma(\tfrac{1}{2} - \nu + \mu)\, [C(\mu) \cos \mu\phi + D(\mu) \sin \mu\phi] \times P_{\nu-\frac{1}{2}}^{-\mu}(\cos \theta)\, \mu d\mu. \quad ...(4.19)$$

Employing the boundary condition (4.15) one obtains from (4.18)

$$A(\mu) = 0,$$

and therefore

$$\bar{\phi}_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tfrac{1}{2} + \nu + \mu)\, \Gamma(\tfrac{1}{2} - \nu + \mu)\, B(\mu) \sin \mu\phi\, P_{\nu-\frac{1}{2}}^{-\mu}(\cos \theta)\, \mu d\mu. \quad ...(4.20)$$

From (4.7), (4.17) and (4.20) one obtains

$$(3 - 4\nu_1)\, C(\mu) = \mu B(\mu). \quad ...(4.21)$$

Again from (4.8), (4.19) and (4.20) it follows that

$$\cos \gamma \sin \mu\gamma \,.\, B(\mu) + \sin \gamma\, [C(\mu) \cos \mu\gamma + D(\mu) \sin \mu\gamma] = 0. \quad ...(4.22)$$

Solving (4.21) and (4.22),

$$D(\mu) = -\frac{B(\mu)}{(3 - 4\nu_1) \sin \mu\gamma \,.\, \sin \gamma} [(3 - 4\nu_1) \sin \mu\gamma \cos \gamma + \mu \cos \mu\gamma \sin \gamma]. \quad ...(4.23)$$

Using the above values of $C(\mu)$ and $D(\mu)$ in (4.21) and (4.23), (4.19) gives

$$\bar{\phi}_2 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(\tfrac{1}{2} + \nu + \mu)\, \Gamma(\tfrac{1}{2} - \nu + \mu)\, B(\mu) \times \left[\frac{\mu \cos \mu\phi}{3 - 4\nu_1} - \frac{(3 - 4\nu_1) \sin \mu\gamma \cos \gamma + \mu \cos \mu\gamma \sin \gamma}{(3 - 4\nu_1) \sin \mu\gamma \sin \gamma} \sin \mu\phi\right] \times P_{\nu-\frac{1}{2}}^{-\mu}(\cos \theta)\, \mu d\mu. \quad ...(4.24)$$

Now from (4.20) and (4.24), on using the boundary conditions (4.11) and (4.12), one gets respectively the following integral equations after some simplification:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} B(\mu)\frac{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}{(3-4\nu_1)\sin\mu\gamma\sin\gamma}[\mu^2\sin^2\gamma-(3-4\nu_1)^2\sin^2\mu\gamma]$$

$$\times P_{\nu-\frac{1}{2}}^{-\mu}(\cos\theta)\,\mu d\mu$$

$$=\begin{cases}\tilde{f}(\theta), & \text{for } 0\leqslant\theta<\alpha,\\ h(\theta), \text{ say}, & \text{for } \alpha<\theta\leqslant\pi\end{cases} \qquad \ldots(4.25)$$

and

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} B(\mu)\frac{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}{(3-4\nu_1)\sin\mu\gamma\sin\gamma}$$

$$\times[(3-4\nu_1)\sin\mu\gamma\cos\mu\gamma+\mu\sin\gamma\cos\gamma]\,P_{\nu-\frac{1}{2}}^{-\mu}(\cos\theta)\,\mu d\mu$$

$$=\begin{cases}g(\theta), \text{ say}, & \text{for } 0\leqslant\theta<\alpha,\\ 0, & \text{for } \alpha<\theta\leqslant\pi.\end{cases} \qquad \ldots(4.26)$$

By applying the Legendre transformation formula in θ-coordinates as is given by Felsen (1958)

$$\Psi(r,\mu,\phi)=\int_0^{\pi}\psi(r,\theta,\phi)\,P_{\nu-\frac{1}{2}}^{-\mu}(-\cos\theta)\frac{d\theta}{\sin\theta}, \qquad \ldots(4.27)$$

$$\psi(r,\theta,\phi)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu)\,\Psi(r,\mu,\phi)$$

$$\times P_{\nu-\frac{1}{2}}^{-\mu}(\cos\theta)\,\mu d\mu,$$

one can invert the integral equations in (4.25) and (4.26) to obtain the following (wo respective results:

$$B(\mu)\,.\,\frac{[(3-4\nu_1)\sin\mu\gamma\cos\mu\gamma+\mu\sin\gamma\cos\gamma]}{(3-4\nu_1)\sin\mu\gamma\sin\gamma}$$

$$=\int_0^{\alpha} g(\theta)\,P_{\nu-\frac{1}{2}}^{-\mu}(-\cos\theta)\frac{d\theta}{\sin\theta}, \qquad \ldots(4.28)$$

and

$$B(\mu)\,.\,\frac{[\mu^2\sin^2\gamma-(3-4\nu_1)^2\sin^2\mu\gamma]}{(3-4\nu_1)\sin\mu\gamma\sin\gamma}$$

$$=\int_0^{\alpha}\tilde{f}(\theta)\,P_{\nu-\frac{1}{2}}^{-\mu}(-\cos\theta)\frac{d\theta}{\sin\theta}+\int_{\alpha}^{\pi}h(\theta)\,P_{\nu-\frac{1}{2}}^{-\mu}(-\cos\theta)\frac{d\theta}{\sin\theta}.$$

$$\ldots(4.29)$$

Eliminating $B(\mu)$ from (4.28) and (4.29),

$$\frac{\mu^2 \sin^2 \gamma - (3 - 4\nu_1)^2 \sin^2 \mu\gamma}{\mu \sin \gamma \cos \gamma + (3 - 4\nu_1) \sin \mu\gamma \cos \mu\gamma} \cdot \int_0^\alpha g(\theta) P_{\nu-\frac{1}{2}}^{-\mu}(-\cos \theta) \frac{d\theta}{\sin \theta}$$

$$= \int_0^\alpha \bar{f}(\theta) P_{\nu-\frac{1}{2}}^{-\mu}(-\cos \theta) \frac{d\theta}{\sin \theta} + \int_\alpha^\pi h(\theta) P_{\nu-\frac{1}{2}}^{-\mu}(-\cos \theta) \frac{d\theta}{\sin \theta}, \quad \ldots(4.30)$$

which gives the desired Fredholm integral equation of second kind when $g(\theta)$, $h(\theta)$ are substituted in terms of integrals from (4.25) and (4.26). This equation will be solved in the next section by Wiener-Hopf technique.

5. Solution of the Integral Equation by Wiener-Hopf Technique

We know that (*cf.* Erdelyi *et al.* 1953)

$$P_{\nu-\frac{1}{2}}^{-\mu}(-\cos \theta)$$

$$= \Gamma(\mu) \{\Gamma(\tfrac{1}{2} + \nu + \mu) \Gamma(\tfrac{1}{2} - \nu + \mu)\}^{-1} \cdot \left(\tan \frac{\theta}{2}\right)^{-\mu}$$

$$\times F\left(-\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; 1 - \mu; \frac{1 - \cos \theta}{2}\right)$$

$$+ \frac{\cos \nu\pi}{\pi} \Gamma(-\mu) \left(\tan \frac{\theta}{2}\right)^{\mu}$$

$$\times F\left(-\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; 1 + \mu; \frac{1 - \cos \theta}{2}\right), \quad \text{for } 0 < \theta < \alpha, \quad \ldots(5.1)$$

and

$$P_{\nu-\frac{1}{2}}^{-\mu}(-\cos \theta)$$

$$= \frac{1}{\Gamma(1 + \mu)} \left(\tan \frac{\theta}{2}\right)^{-\mu} F\left(-\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; 1 + \mu; \frac{1 + \cos \theta}{2}\right),$$

$$\text{for } \alpha < \theta < \pi. \quad \ldots(5.2)$$

Let

$$X(\mu) = \left(\tan \frac{\alpha}{2}\right)^{\mu} \int_0^\alpha g(\theta) \left(\tan \frac{\theta}{2}\right)^{-\mu}$$

$$\times F\left(-\nu + \tfrac{1}{2}, \nu + \tfrac{1}{2}; 1 - \mu; \frac{1 - \cos \theta}{2}\right) \frac{d\theta}{\sin \theta}, \quad \ldots(5.3)$$

$$Y(\mu) = \Gamma(1 + \mu) \left(\tan \frac{\alpha}{2}\right)^{\mu} \int_\alpha^\pi h(\theta) P_{\nu-\frac{1}{2}}^{-\mu}(-\cos \theta) \frac{d\theta}{\sin \theta}, \quad \ldots(5.4)$$

and

$$Z(\mu) = \left(\tan\frac{\alpha}{2}\right)^{\mu} \int_0^{\alpha} \bar{f}(\theta)\left(\tan\frac{\theta}{2}\right)^{-\mu}$$

$$\times F\left(-\nu+\tfrac{1}{2},\ \nu+\tfrac{1}{2};\ 1-\mu;\ \frac{1-\cos\theta}{2}\right)\frac{d\theta}{\sin\theta}. \qquad \ldots(5.5)$$

Also the behaviour of $P_{\nu-\frac{1}{2}}^{-\mu}(-\cos\theta)$ near its singularity $\theta=\pi$ is given by

$$P_{\nu-\frac{1}{2}}^{-\mu}(-\cos\theta) \sim \frac{2^{-\mu/2}(1+\cos\theta)^{\mu/2}}{\Gamma(1+\mu)}$$

and therefore, if one assumes at the edge [at $\theta=\pi$] the condition

$$h(\theta) \sim (1+\cos\theta)^{\epsilon}, \quad \text{as} \quad \theta \to \pi, \qquad \ldots(5.6)$$

where ϵ is small and positive, then $Y(\mu)$ is a regular function of μ in the complex μ-plane for $\operatorname{Re}\mu > -2\epsilon$ and $Y(\mu) \sim 1/|\mu|$, as $|\mu| \to \infty$, $\operatorname{Re}\mu > -2\epsilon$. Also from (5.3) and (5.5) it is apparent that $X(\mu)$ and $Z(\mu)$ are regular functions of μ in the left half of the complex μ-plane for $\operatorname{Re}\mu < 2\epsilon$ and each of them are of the order $1/|\mu|$, as $|\mu| \to \infty$, $\operatorname{Re}\mu < 2\epsilon$, if one assumes the edge [on $\theta=0$] conditions

$$g(\theta) \sim (1-\cos\theta)^{\epsilon}, \qquad \ldots(5.7)$$

and

$$\bar{f}(\theta) \sim (1-\cos\theta)^{\epsilon}, \qquad \ldots(5.8)$$

as $\theta \to 0$, where ϵ is a small and positive quantity.

Using these definitions of $X(\mu)$, $Y(\mu)$ and $Z(\mu)$, the integral equation (4.30) leads to the following equation

$$K(\mu)\left[\frac{\Gamma(\mu)\,\Gamma(1+\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}X(\mu) + \frac{\cos\nu\pi}{\pi}\left(\tan\frac{\alpha}{2}\right)^{2\mu}\right.$$

$$\left.\times\,\Gamma(1+\mu)\,\Gamma(-\mu)\,X(-\mu)\right]$$

$$= Y(\mu) + \left[\frac{\Gamma(\mu)\,\Gamma(1+\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}Z(\mu) + \frac{\cos\nu\pi}{\pi}\left(\tan\frac{\alpha}{2}\right)^{2\mu}\right.$$

$$\left.\times\,\Gamma(1+\mu)\,\Gamma(-\mu)\,Z(-\mu)\right], \qquad \ldots(5.9)$$

where

$$K(\mu) = \frac{2\,[\mu^2\sin^2\gamma - (3-4\nu_1)^2\sin^2\mu\gamma]}{\mu\sin 2\gamma + (3-4\nu_1)\sin 2\mu\gamma}. \qquad \ldots(5.10)$$

After little rearrangement (5.9) is

$$K(\mu)\Big[X(\mu)+\frac{\cos\nu\pi}{\pi}\Big(\tan\frac{\alpha}{2}\Big)^{2\mu}\frac{\Gamma(-\mu)}{\Gamma(\mu)}$$
$$\times\,\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu)\,X(-\mu)\Big]$$
$$=\frac{\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu)}{\Gamma(\mu)\,\Gamma(1+\mu)}\,Y(\mu)$$
$$+\Big[Z(\mu)+\frac{\cos\nu\pi}{\pi}\Big(\tan\frac{\alpha}{2}\Big)^{2\mu}\frac{\Gamma(-\mu)}{\Gamma(\mu)}$$
$$\times\,\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu)\,Z(-\mu)\Big]. \qquad ...(5.11)$$

Assuming now, the possibility of expansion of $K(\mu)$ into the form

$$K(\mu)=K_-(\mu)/K_+(\mu), \qquad ...(5.12)$$

where $K_-(\mu)$ and $K_+(\mu)$ are respectively regular in the left half and right half of the complex μ-plane and both of them are of the finite order for $|\mu|\gg 1$, equation (5.11) takes the form

$$K_-(\mu)\,X(\mu)+\frac{\cos\nu\pi}{\pi}\Big(\tan\frac{\alpha}{2}\Big)^{2\mu}\frac{\Gamma(-\mu)}{\Gamma(\mu)}\,\Gamma(\tfrac{1}{2}+\nu+\mu)$$
$$\times\,\Gamma(\tfrac{1}{2}-\nu+\mu)\,K_-(\mu)\,X(-\mu)$$
$$=\frac{\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu)}{\Gamma(\mu)\,\Gamma(1+\mu)}\,K_+(\mu)\,Y(\mu)+Z(\mu)\,K_+(\mu)$$
$$+\frac{\cos\nu\pi}{\pi}\Big(\tan\frac{\alpha}{2}\Big)^{2\mu}\frac{\Gamma(-\mu)}{\Gamma(\mu)}\,\Gamma(\tfrac{1}{2}+\nu+\mu)$$
$$\times\,\Gamma(\tfrac{1}{2}-\nu+\mu)\,K_+(\mu)\,Z(-\mu). \qquad ...(5.13)$$

In studying eqn. (5.13) one can draw the following conclusions about its different terms. The first term on its left-hand side is regular for $\mathrm{Re}\,\mu<2\epsilon$ and it tends to zero for $|\mu|\gg 1$ in its region of regularity and the second term there also tends to zero for $|\mu|\gg 1$ and is regular for $\mathrm{Re}\,\mu>-2\epsilon$ except at the poles of $\Gamma(-\mu)$ and $K_-(\mu)$. However, the pole at $\mu=0$ of $\Gamma(-\mu)$ is not a point of singularity of the term as it contains $\Gamma(\mu)$ in its denominator. The first term on the right-hand side is regular for $\mathrm{Re}\,\mu>-2\epsilon$ and vanishes there for $|\mu|\gg 1$. The second term there is regular for $\mathrm{Re}\,\mu<2\epsilon$ except at the poles of $K_+(\mu)$ and it tends to zero for $|\mu|\gg 1$ in its region of regularity. The third term is regular for $\mathrm{Re}\,\mu>-2\epsilon$ except at the poles $\mu=1,2,3,\ldots$ of $\Gamma(-\mu)$ and it tends to zero there for $|\mu|\gg 1$. In the above analysis α is assumed to be less than $\pi/2$.

The second term on the left-hand side of (5.13) can be decomposed in the following manner:

$$\frac{\cos\nu\pi}{\pi}\Big(\tan\frac{\alpha}{2}\Big)^{2\mu}\frac{\Gamma(-\mu)}{\Gamma(\mu)}\,\Gamma(\tfrac{1}{2}+\nu+\mu)\,\Gamma(\tfrac{1}{2}-\nu+\mu)\,K_-(\mu)\,X(-\mu)$$
$$=H_++\Sigma_{1-}+\Sigma_{2-}, \qquad ...(5.14)$$

where

$$\sum_{1-} = \sum_{m=1}^{\infty} a_m \left(\tan \frac{a}{2}\right)^{2m} \frac{X(-m)}{\mu - m}, \qquad \dots(5.15)$$

$$\sum_{2-} = \sum_{m=1}^{\infty} b_m \left(\tan \frac{a}{2}\right)^{2\mu'_m} \frac{X(-\mu'_m)}{\mu - \mu'_m}, \qquad \dots(5.16)$$

$$a_m = \frac{\cos \nu\pi}{\pi} \frac{(-1)^{m+1}}{m!\,\Gamma(m)} \Gamma(\tfrac{1}{2} + \nu + m)\,\Gamma(\tfrac{1}{2} - \nu + m)\,K_-(m), \qquad \dots(5.17)$$

$$b_m = \frac{\cos \nu\pi}{\pi} \frac{\Gamma(-\mu'_m)}{\Gamma(+\mu'_m)} \Gamma(\tfrac{1}{2} + \nu + \mu'_m)\,\Gamma(\tfrac{1}{2} - \nu + \mu'_m)$$

$$\times \left\{ \lim_{\mu \to \mu'_m} [(\mu - \mu'_m)\,K_-(\mu)] \right\}, \qquad \dots(5.18)$$

$$H_+ = \text{l.h.s. of } (5.14) - \Sigma_{1-} - \Sigma_{2-}, \qquad \dots(5.19)$$

and μ'_m, $m = 1, 2, 3, \dots$ etc., are the poles of $K_-(\mu)$ in the half plane $\text{Re}\,\mu > -2\epsilon$.

The second term on the right-hand side of (5.13) can be decomposed as

$$Z(\mu)\,K_+(\mu) = P_-(\mu) + P_+(\mu), \qquad \dots(5.20)$$

where

$$P_+(\mu) = \sum_{m=1}^{\infty} Z(-\mu_m) \frac{1}{\mu + \mu_m} \left\{ \lim_{\mu \to -\mu_m} [(\mu + \mu_m)\,K_+(\mu)] \right\}, \qquad \dots(5.21)$$

$$P_-(\mu) = \text{l.h.s. of } (5.20) - P_+(\mu), \qquad \dots(5.22)$$

and $\mu = -\mu_m$ are the poles of $K_+(\mu)$ in the half plane $\text{Re}\,\mu < 2\epsilon$.

Again, the third term on the right-hand side of (5.13) can be decomposed as

$$\frac{\cos \nu\pi}{\pi} \left(\tan \frac{a}{2}\right)^{2\mu} \frac{\Gamma(-\mu)}{\Gamma(\mu)} \Gamma(\tfrac{1}{2} + \nu + \mu)\,\Gamma(\tfrac{1}{2} - \nu + \mu)\,K_+(\mu)\,Z(-\mu)$$

$$= Q_+(-\mu) + Q_-(-\mu), \qquad \dots(5.23)$$

where

$$Q_-(-\mu) = \sum_{m=1}^{\infty} d_m \left(\tan \frac{a}{2}\right)^{2m} \frac{Z(-m)}{\mu - m}, \qquad \dots(5.24)$$

$$d_m = \frac{\cos \nu\pi}{\pi} \frac{(-1)^{m+1}}{m!\,\Gamma(m)} \Gamma(\tfrac{1}{2} + \nu + m)\,\Gamma(\tfrac{1}{2} - \nu + m)\,K_+(m), \qquad \dots(5.25)$$

and

$$Q_+(-\mu) = \text{l.h.s. of } (5.23) - Q_-(-\mu). \qquad \dots(5.26)$$

Using the above results in (5.14) to (5.26), the equation (5.13) can be expressed as

$$K_-(\mu)\,X(\mu)+\Sigma_{1-}+\Sigma_{2-}-P_-(\mu)-Q_-(-\mu)$$
$$=-H_++P_+(\mu)+Q_+(-\mu)+\frac{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}{\Gamma(\mu)\,\Gamma(1+\mu)}$$
$$\times K_+(\mu)\,Y(\mu). \qquad ...(5.27)$$

The left and right sides of the above equation are regular in the half planes $\text{Re}\,\mu<2\epsilon$ and $\text{Re}\,\mu>-2\epsilon$ respectively and both sides vanish for $|\mu|\gg 1$. So by Liouville's theorem

$$K_-(\mu)\,X(\mu)+\Sigma_{1-}+\Sigma_{2-}-P_-(\mu)-Q_-(-\mu)=0,$$
$$-H_++P_+(\mu)+Q_+(-\mu)+\frac{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}{\Gamma(\mu)\,\Gamma(1+\mu)}$$
$$\times K_+(\mu)\,Y(\mu)=0. \qquad ...(5.28)$$

The first of the above equations in (5.28) determines $X(\mu)$. As the zeros of the numerator and denominator of $K(\mu)$ in (5.10) are in general complex, the desired type of its decomposition [*cf.* (5.12)] will lead to a very heavy calculation. Therefore, to overcome this difficulty one can replace it with a sufficient fair approximation, as suggested by Koiter (1954), by

$$K(\mu)\approx 2\mu K^*(\mu)=2\mu\left[\frac{a}{\mu\gamma}\tan b\mu\gamma\right], \qquad ...(5.29)$$

where a and b are suitable constants. Then applying the factor theorem (*cf.* Noble 1958) one can write

$$2\mu K^*(\mu)=2\mu ab\left[\frac{\sin b\mu\gamma}{b\mu\gamma}\Big/\cos b\mu\gamma\right]$$
$$=2\mu ab\,\frac{K_-^*(\mu)}{K_+^*(\mu)}, \qquad ...(5.30)$$

where

$$K_-^*(\mu)=\frac{1}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{1}{2}-\frac{b\mu\gamma}{\pi}\right)}{\Gamma\left(1-\frac{b\mu\gamma}{\pi}\right)},$$

$$K_+^*(\mu)=\sqrt{\pi}\,\frac{\Gamma\left(1+\frac{b\mu\gamma}{\pi}\right)}{\Gamma\left(\frac{1}{2}+\frac{b\mu\gamma}{\pi}\right)}=\frac{1}{K_-^*(-\mu)}. \qquad ...(5.31)$$

Also, for $|\mu|\gg 1$,

$$K_-^*(\mu)\sim|\mu|^{-1/2},$$
$$K_+^*(\mu)\sim|\mu|^{1/2},$$

and therefore,

$$K_-(\mu) \approx 2\mu ab\, K_-^*(\mu) \sim |\mu|^{1/2},$$

$$K_+(\mu) \approx K_+^*(\mu) \sim |\mu|^{1/2}. \qquad ...(5.32)$$

With the above approximations eqns. (5.16) to (5.18), (5.21), (5.22), (5.24) and (5.25) respectively are

$$\sum_{2-} = \sum_{m=0}^{\infty} b_m \left(\tan\frac{\alpha}{2}\right)^{\pi(2m+1)/b\gamma} X\left(-\frac{\pi}{b\gamma}\frac{2m+1}{2}\right) \frac{1}{\mu - \frac{\pi}{b\gamma}\frac{2m+1}{2}}, \qquad ...(5.33)$$

$$a_m = \frac{2\cos\nu\pi}{\pi}\frac{(-1)^{m+1}}{m!\,\Gamma(m)}\Gamma(\tfrac{1}{2}+\nu+m)\,\Gamma(\tfrac{1}{2}-\nu+m)\,K_-^*(m)\,\mu ab, \qquad ...(5.34)$$

$$b_m = \frac{\cos\nu\pi}{\pi}\,\frac{\Gamma\left(-\frac{\pi}{b\gamma}\frac{2m+1}{2}\right)}{\Gamma\left(\frac{\pi}{b\gamma}\frac{2m+1}{2}\right)} \times \frac{\Gamma\left(\frac{1}{2}+\nu+\frac{\pi}{b\gamma}\frac{2m+1}{2}\right)\Gamma\left(\frac{1}{2}-\nu+\frac{\pi}{b\gamma}\frac{2m+1}{2}\right).\,(-1)^m}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-m\right).\,m!}, \qquad ...(5.35)$$

$$P_+(\mu) = \sum_{m=0}^{\infty} C_m \frac{Z\left(-\frac{\pi}{b\gamma}(m+1)\right)}{\mu + \frac{\pi}{b\gamma}(m+1)}; \quad C_m = \frac{(-1)^m\sqrt{\pi}}{m!\,\Gamma(-m-\frac{1}{2})}, \qquad ...(5.36)$$

where

$$P_-(\mu) = Z(\mu)\,K_+^*(\mu) - P_+(\mu), \qquad ...(5.37)$$

$$Q_-(\mu) = \sum_{m=1}^{\infty} d_m \left(\tan\frac{\alpha}{2}\right)^{2m} \frac{Z(-m)}{\mu - m}, \qquad ...(5.38)$$

and

$$d_m = \frac{(-1)^m}{m!\,\Gamma(m)}\frac{\cos\nu\pi}{\pi}\Gamma(\tfrac{1}{2}+\nu+m)\,\Gamma(\tfrac{1}{2}-\nu+m)\,K_+^*(m). \qquad ...(5.39)$$

Using the above results in (5.33) to (5.39), the first equation of (5.28) gives

$$2\mu ab\, K_-^*(\mu)\, X(\mu) + \sum_{m=1}^{\infty} a_m \left(\tan\frac{\alpha}{2}\right)^{2m} \frac{X(-m)}{\mu - m}$$

$$+ \sum_{m=0}^{\infty} b_m \left(\tan\frac{\alpha}{2}\right)^{\pi(2m+1)/b\gamma} X\left(-\frac{\pi(2m+1)}{2b\gamma}\right) . \frac{1}{\mu - \frac{\pi}{b\gamma}\frac{2m+1}{2}} -$$

(equation continued on p. 153)

$$- K_+^* (\mu) Z(\mu) + \sum_{m=0}^{\infty} C_m Z\left(-\frac{\pi}{b\gamma}(m+1)\right) \frac{1}{\mu + \frac{\pi}{b\gamma}(m+1)}$$

$$- \sum_{m=1}^{\infty} d_m \left(\tan\frac{\alpha}{2}\right)^{2m} \frac{Z(-m)}{\mu - m} = 0. \qquad \ldots(5.40)$$

Let $\pi/b\gamma = p/q$, where p, q are integers. Then writing

$$\left(\tan\frac{\alpha}{2}\right)^{1/q} = x, \quad 2\mu ab\, K_-^* (\mu) X(\mu) = R(\mu)$$

and

$$K_+^* (\mu) Z(\mu) = S(\mu),$$

the eqn. (5.40) is

$$R(\mu) + \sum_{m=1}^{\infty} \frac{a_m}{-2mab\, K_-^*(-m)} \frac{R(-m)}{\mu - m} x^{2mq}$$

$$+ \sum_{m=0}^{\infty} \frac{b_m}{K_-^*\left(-\frac{p}{q}\frac{2m+1}{2}\right)} \frac{R\left(-\frac{p}{q}\frac{2m+1}{2}\right)}{\mu - \frac{p}{q}\frac{2m+1}{2}} \frac{x^{(2m+1)p}}{-2ab\frac{p}{q}\cdot\frac{2m+1}{2}}$$

$$- S(\mu) + \sum_{m=0}^{\infty} \frac{c_m}{K_+^*\left(-\frac{p}{q}(m+1)\right)} \frac{S\left(-\frac{p}{q}(m+1)\right)}{\mu + \frac{p}{q}(m+1)}$$

$$- \sum_{m=1}^{\infty} \frac{d_m}{K_+^*(-m)} \frac{S(-m)}{\mu - m} x^{2mq} = 0. \qquad \ldots(5.41)$$

From (5.41) it is clear that $R(\mu)$ can be expanded as a series in the form

$$R(\mu) = S(\mu) + \sum_{n=0}^{\infty} R_n(\mu) x^n. \qquad \ldots(5.42)$$

Equation (5.41) under (5.42) is

$$\sum_{n=0}^{\infty} R_n(\mu) x^n + \sum_{m=1}^{\infty} \bar{a}_m \left\{S(-m) + \sum_{n=0}^{\infty} R_n(-m) x^n\right\} \frac{x^{2mq}}{\mu - m}$$

$$+ \sum_{m=0}^{\infty} \bar{b}_m \left\{S\left(-\frac{p}{q}\frac{2m+1}{2}\right) + \sum_{n=0}^{\infty} R_n\left(-\frac{p}{q}\frac{2m+1}{2}\right) x^n\right\} \times$$

(equation continued on p. 154)

$$\times \frac{x^{(2m+1)p}}{\mu - \frac{p}{q}\frac{2m+1}{2}} + \sum_{m=0}^{\infty} \bar{c}_m \frac{S\left(-\frac{p}{q}(m+1)\right)}{\mu + \frac{p}{q}(m+1)}$$

$$-\sum_{m=1}^{\infty} \bar{d}_m \frac{S(-m)}{\mu - m} x^{2mq} = 0, \qquad \ldots(5.43)$$

where

$$\bar{a}_m = \frac{a_m}{-2mab\,K_-^*(-m)}, \quad \bar{b}_m = \frac{b_m}{-2ab\frac{p}{q}\frac{2m+1}{2}\,K_-^*\left(-\frac{p}{q}\frac{2m+1}{2}\right)},$$

... etc.

Equating coefficients of like powers of x from both sides of (5.43), the unknowns $R_0(\mu)$, $R_1(\mu)$, $R_2(\mu)$, ... etc., can be determined. Thus $R(\mu)$ and hence $X(\mu)$ and $X(-\mu)$ can be obtained. Therefore, from (4.28), $B(\mu)$ is known and consequently $C(\mu)$ and $D(\mu)$ are obtained from (4.21) and (4.23) respectively. Then from (4.20) and (4.24), after taking their Mellin inversions, ϕ_1 and ϕ_2 are determined and, therefore, the stresses and displacements at any point of the solid can be calculated. Thus the solution of the problem is complete.

As a particular case of the problem, one can consider the Poisson's ratio $\nu_1 = \frac{1}{4}$ of the elastic material of the wedge having the wedge angle $\gamma = \pi/4$. Then from (5.29), one gets

$$a \approx -0{\cdot}16$$
$$b \approx 3{\cdot}00.$$

Also, since

$$\frac{\pi}{b\gamma} = \frac{4}{3} = \frac{p}{q}, \quad p = 4 \quad \text{and} \quad q = 3.$$

Then (5.43) is

$$\sum_{n=0}^{\infty} R_n(\mu)\,x^n + \sum_{m=1}^{\infty} \bar{a}_m \left\{S(-m) + \sum_{n=0}^{\infty} R_n(-m)\,x^n\right\} \frac{x^{6m}}{\mu - m}$$

$$+ \sum_{m=0}^{\infty} \bar{b}_m \left\{S\left(-\frac{4}{3}\frac{2m+1}{2}\right) + \sum_{n=0}^{\infty} R_n\left(-\frac{4}{3}\frac{2m+1}{2}\right)x^n\right\}$$

$$\times \frac{x^{4(2m+1)}}{\mu - \frac{4}{3}\frac{2m+1}{2}} + \sum_{m=0}^{\infty} \bar{c}_m \frac{S\left(-\frac{4}{3}(m+1)\right)}{\mu + \frac{4}{3}(m+1)}$$

$$-\sum_{m=1}^{\infty} \bar{d}_m \frac{S(-m)}{\mu - m} x^{6m} = 0. \qquad \ldots(5.44)$$

By equating terms of like powers of x from both sides of (5.44) now one obtains

$$R_0(\mu) = -\sum_{m=0}^{\infty} c_m \frac{S\left(-\frac{4}{3}(m+1)\right)}{\mu + \frac{4}{3}(m+1)},$$

$$R_1(\mu) = R_2(\mu) = R_3(\mu) = 0.$$

$$R_4(\mu) = -\bar{b}_0 \left[S\left(-\frac{2}{3}\right) - \sum_{m=0}^{\infty} \bar{c}_m \frac{S\left(-\frac{4}{3}(m+1)\right)}{-\frac{2}{3} + \frac{4}{3}(m+1)} \right] \cdot \frac{1}{\mu - \frac{2}{3}},$$

$$R_5(\mu) = 0,$$

$$R_6(\mu) = -\bar{a}_1 \left[S(-1) - \sum_{m=0}^{\infty} \bar{c}_m \frac{S\left(-\frac{4}{3}(m+1)\right)}{-1 + \frac{4}{3}(m+1)} \right] \frac{1}{\mu - 1}$$

$$+ \bar{d}_1 \frac{S(-1)}{\mu - 1}, \quad \text{etc.}$$

Thus $R_n(\mu)$, $n = 0, 1, 2, 3, \ldots$ is determined and hence ϕ_1 and ϕ_2 can be determined after performing four simple integrations as are given in (4.20), (4.24) and (4.2). Numerical computations of these single integrals can also be done up to any desired degree of accuracy by using a digital computer.

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References

Aleksandrov, V. M., and Chebakov, M. I. (1972). Mixed problems of the mechanics of continuous media associated with Hankel transform and Mehler-Fok integral transforms. *P.M.M.*, **36**, 494.

Erdelyi, A. *et al.* (1953). Higher Transcendental Functions, Vol. I. McGraw-Hill Book Co., Inc., New York, 143.

Felsen, L. B. (1958). Some new transform theorems involving Legendre functions. *J. Math. Phys.*, **37**, 155.

Jones, D. S. (1950). Note on diffraction by an edge. *Q. Jl Mech. appl. Math.*, **3**, 420.

Koiter, W. T. (1954). Approximate solution of Wiener-Hopf type integral equations with applications, Parts I–III. *Proc. Kon. Ned. Acad. Wet. B*, **57**, 558.

Low, R. D. (1966). On the torsion of an elastic cone as a mixed boundary value problem. *Q. Jl Mech. appl. Math.*, **19**, 57.

Lur'e, A. I. (1964). Three Dimensional Problems of the Theory of Elasticity. Interscience Publishers, New York.

Matczynski, M. (1963). Elastic wedge with discontinuous boundary conditions. *Arch. Mech. Stos.*, **15**, 833.

Mandal, B. (1975). Diffraction of the field due to a point source in a three-dimensional region bounded by two planes and the two convex surfaces of a sphere. (In press).

Noble, B. (1958). Methods Based on Wiener-Hopf Technique. Pergamon Press, New York.

Nuller, B. M. (1970). Contact problem for an elastic infinite cone. *P.M.M.*, **34**, 2, 339.

Pridmore-Brown, D. C. (1968). A Wiener-Hopf solution of a radiation problem in conical geometry. *J. Math. Phys.*, **47**, 79.