ON A CLASS OF STARLIKE FUNCTIONS IN THE UNIT DISC II

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A representation formula, distortion theorems and the radius of convexity are determined for the class $S_q(\alpha)$ of analytic functions whose power series begins $f(z) = z + a_{q-1}z^{q+1} + \cdots$ and satisfying the condition $|(zf'(z)/f(z) - 1)/(zf'(z)/f(z) + 1)| < \alpha$, for some $\alpha\,(0 < \alpha \leq 1)$ and for all z in $E \equiv \{z: |z| < 1\}$. The results extend the corresponding work of Padmanabhan. A sufficient condition for a function to belong to $S_q(\alpha)$ has also been obtained. Moreover we use a different technique to determine the radius of convexity.

1. INTRODUCTION

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc E. If $f(z)$ satisfies the condition

$$\text{Re}\left\{ z\frac{f'(z)}{f(z)}\right\} > 0 \qquad \ldots(1\cdot1)$$

for all z in E, then it is well known (see for example Polya and Szegō 1964, p. 105, Problem 109) that (1·1) is both necessary and sufficient for $f(z)$ to be univalent and starlike in the unit disc E. For starlike functions, various definitions of 'order' have been used by different workers in the field (see e.g. Libra 1964, Robertson 1936 and others). Recently, Padmanabhan (1968) has introduced the 'concept' of 'order' of starlikeness in a different manner. Thus, according to him, A function $f(z)$ analytic in the unit disc E, normalized by $f(0) = 0$, $f'(0) = 1$ and satisfying for all z in E the condition

$$\left|\left(z\frac{f'(z)}{f(z)} - 1\right)\Big/\left(z\frac{f'(z)}{f(z)} + 1\right)\right| < \alpha, \qquad \ldots(1\cdot2)$$

for a given $\alpha\,(0 < \alpha \leq 1)$ is said to be starlike of order α in E. We denote the class of all such functions by $S(\alpha)$. For the class $S(\alpha)$, Padmanabhan (1968) has obtained representation formula, distortion theorems and the radius of convexity. The author (Mogra 1976) obtained coefficient estimates and a sufficient condition for functions in $S(\alpha)$, the analogue of which has not been obtained by Padmanabhan (1968).

In this paper we consider functions analytic in the unit disc whose power series begins

$$f(z) = z + a_{q+1}z^{q+1} + a_{q+2}z^{q+2} + \cdots \qquad \ldots(1\cdot3)$$

and which are starlike of order α in the sense of Padmanabhan. The class of such functions we denote by $S_q(\alpha)$. We obtain a representation formula, distortion theorems, radius of convexity, etc., for $f \in S_q(\alpha)$. For $q = 1$, our results yield the corresponding results obtained by Padmanabhan (1968) and Mogra (1976).

2. A Representation Formula

Let $B_\alpha (0 < \alpha \leq 1)$ denote the class of functions $\psi(z)$ which are analytic in the unit disc E and satisfy $|\psi(z)| \leq \alpha$ for all $z \in E$. We obtain representation formula for functions in $S_q(\alpha)$ in terms of functions in B_α. We require the following lemma.

Lemma 1—Let $H(z) = 1 + h_q z^q + \ldots$ Then $H(z)$ is analytic and satisfies the condition

$$|(1 - H(z))/(1 + H(z))| < \alpha (0 < \alpha \leq 1)$$

for $|z| < 1$ if and only if there exists a function $\psi(z) \in B_\alpha$ such that

$$H(z) = (1 - z^q \psi(z))/(1 + z^q \psi(z)).$$

The Lemma 1 follows exactly on the same lines as those of Padmanabhan (1968, Lemma 1); so we omit the proof.

Theorem 1 — Let $f(z) = z + \sum_{k=1}^{\infty} a_{q+k} z^{q+k}$ be analytic in the unit disc E.

Then $f(z) \in S_q(\alpha)$ if and only if

$$f(z) = z \exp\left\{-2 \int_0^z \frac{t^{q-1}\psi(t)}{1 + t^q \psi(t)} dt\right\} \quad \ldots(2 \cdot 1)$$

for some $\psi(z) \in B_\alpha$.

Proof: Let $f(z) \in S_q(\alpha)$, it is easily seen that $zf'(z)/f(z)$ satisfies the hypothesis of the Lemma 1. Hence there exists $\psi(z) \in B_\alpha$ such that

$$z \frac{f'(z)}{f(z)} = \frac{1 - z^q \psi(z)}{1 + z^q \psi(z)}.$$

Thus, we have

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{-2z^{q-1}\psi(z)}{1 + z^q \psi(z)}. \quad \ldots(2 \cdot 2)$$

Integration gives (2·1) easily. Conversely, if $f(z)$ has the representation (2·1) for some $\psi(z) \in B_\alpha$; then, it follows that

$$z \frac{f'(z)}{f(z)} = \frac{1 - z^q \psi(z)}{1 + z^q \psi(z)}.$$

Hence Lemma 1 gives that $f(z) \in S_q(\alpha)$. Hence the theorem.

3. Distortion Theorems

Theorem 2—Let $f(z) = z + \sum_{k=1}^{\infty} a_{q+k} z^{q+k}$ be analytic in the unit disc E and suppose $f(z) \in S_q(\alpha)$. Then we have, for $z \in E$,

$$|f(z)| \leq \frac{|z|}{(1 - \alpha |z|^q)^{2/q}} \qquad \ldots(3.1)$$

$$|f(z)| \geq \frac{|z|}{(1 + \alpha |z|^q)^{2/q}} \qquad \ldots(3.2)$$

Both estimates are sharp.

Proof: Since $f(z) \in S_q(\alpha)$, we have, by (2.1)

$$z \frac{f'(z)}{f(z)} = \frac{1 - z^q \psi(z)}{1 + z^q \psi(z)} \qquad \ldots(3.3)$$

for some $\psi(z) \in B_\alpha$. We can write (3.3) as

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{-2z^{q-1} \psi(z)}{1 + z^q \psi(z)}. \qquad \ldots(3.4)$$

Since $\psi(z) \in B_\alpha$, (3.4) gives

$$\log\left(\left|\frac{f(z)}{z}\right|\right) = \mathrm{Re}\left(\log\left(\frac{f(z)}{z}\right)\right)$$

$$= \mathrm{Re} \int_0^z \left[\frac{f'(s)}{f(s)} - \frac{1}{s}\right] ds$$

$$= \mathrm{Re} \int_0^z \frac{-2s^{q-1} \psi(s)}{1 + s^q \psi(s)} ds$$

$$\leq \int_0^{|z|} \frac{2|\psi(te^{i\theta})| t^{q-1}}{|1 + t^q e^{iq\theta} \psi(te^{i\theta})|} dt$$

$$\leq 2 \int_0^{|z|} \frac{\alpha t^{q-1}}{1 - \alpha t^q} dt$$

$$= -\log(1 - \alpha |z|^q)^{2/q}.$$

Thus

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{(1 - \alpha |z|^q)^{2/q}}$$

which gives (3.1). To prove (3.2), we observe that the condition (1.2) coupled with an application of Schwarz lemma implies that, for $|z| < 1$, $zf'(z)/f(z)$ assumes values lying in the open disc E on the line segment joining the points $(1-\alpha|z|^q)/(1+\alpha|z|^q)$ and $(1+\alpha|z|^q)/(1-\alpha|z|^q)$ as diameter. Hence we have

$$\frac{1-\alpha|z|^q}{1+\alpha|z|^q} \leq \operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \leq \frac{1+\alpha|z|^q}{1-\alpha|z|^q}. \qquad ...(3.5)$$

Let $|z| = r$, then (3.5) gives

$$r \operatorname{Re}\left\{\frac{\partial}{\partial r}\left(\log\frac{f(z)}{z}\right)\right\} = \operatorname{Re}\left\{z\frac{f'(z)}{f(z)} - 1\right\}$$

$$\geq \frac{-2\alpha r^q}{1+\alpha r^q}.$$

Thus we have

$$\log\left|\frac{f(z)}{z}\right| = \operatorname{Re}\left\{\log\frac{f(z)}{z}\right\}$$

$$\geq \int_0^r \frac{-2\alpha s^{q-1}}{1+\alpha s^q}\,ds$$

$$= -\log(1+\alpha r^q)^{2/q}.$$

Hence

$$|f(z)| \geq \frac{|z|}{(1+\alpha|z|^q)^{2/q}}.$$

Equality in (3.1) and (3.2) holds for the function

$$f(z) = \frac{z}{(1-\alpha z^q)^{2/q}}.$$

4. A Sufficient Condition for a Function to Belong to $S_q(\alpha)$

Theorem 3—Let $f(z) = z + \sum_{k=1}^{\infty} a_{q+k} z^{q+k}$ be analytic in the unit disc E. If for some $\alpha\,(0 < \alpha \leq 1)$,

$$\sum_{k=1}^{\infty}\left\{\left(\frac{1+\alpha}{2\alpha}\right)(q+k) + \frac{\alpha-1}{2\alpha}\right\}|a_{q+k}| \leq 1, \qquad ...(4.1)$$

then $f(z)$ belongs to $S_q(\alpha)$.

Proof: We employ the technique used by Clunie and Keogh (1960). Thus suppose that (4.1) holds and that

$$f(z) = z + \sum_{k=1}^{\infty} a_{q+k} z^{q+k};$$

then in $|z| < 1$,

$$
\begin{aligned}
&|zf'(z) - f(z)| - \alpha\,|zf'(z) + f(z)| \\
&= \Big|\sum_{k=1}^{\infty} (q+k-1)\, a_{q+k} z^{q+k}\Big| - \alpha\,\Big|2z + \sum_{k=1}^{\infty} (q+k+1)\, a_{q+k} z^{q+k}\Big| \\
&\leq \sum_{k=1}^{\infty} (q+k-1)\,|a_{q+k}|\, r^{q+k} - \alpha\left\{2r - \sum_{k=1}^{\infty} (q+k+1)\,|a_{q+k}|\, r^{q+k}\right\} \\
&< \left[\sum_{k=1}^{\infty} (q+k-1)\,|a_{q+k}| - 2\alpha + \sum_{k=1}^{\infty} \alpha\,(q+k+1)\,|a_{q+k}|\right] r \\
&= \left[\sum_{k=1}^{\infty} \{(1+\alpha)(q+k) + (\alpha-1)\}\,|a_{q+k}| - 2\alpha\right] r \\
&\leq 0.
\end{aligned}
$$

Hence it follows that

$$\left(\frac{zf'(z)}{f(z)} - 1\right)\Big/\left(z\,\frac{f'(z)}{f(z)} + 1\right) < \alpha,$$

therefore $f(z) \varepsilon S_q(\alpha)$.

5. The Radius of Convexity for Functions in the Class $S_q(\alpha)$

Let D denote the class of analytic functions $\omega(z)$ in $|z| < 1$ which satisfy the conditions (i) $\omega(0) = 0$ and (ii) $|\omega(z)| < 1$ for $|z| < 1$. For obtaining the radius of convexity for functions in the class $S_q(\alpha)$, we require the following lemmas.

Lemma 2—If $\omega(z) \in D$, then for $|z| < 1$,

$$|z\omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2}. \qquad \ldots(5.1)$$

This lemma is due to Singh and Goel (1971).

Lemma 3—For $\omega(z) \in D$, we have

$$\operatorname{Re}\left\{\frac{z^q\,\omega'(z) + (q-1)\,z^{q-1}\,\omega(z)}{\left(1 + \alpha z^{q-1}\omega(z)\right)\left(1 - \alpha z^{q-1}\omega(z)\right)}\right\} \leq -\frac{q}{4\alpha^2}\operatorname{Re}\left\{\alpha p(z) - \frac{\alpha}{p(z)}\right\}$$

$$+ \frac{r^{2q}\,|\alpha p(z) + \alpha|^2 - |1 - p(z)|^2}{4\alpha^2 r^{q-1}(1 - r^2)\,|p(z)|} \qquad \ldots(5.2)$$

where

$$p(z) = \left(1 - \alpha z^{q-1}\omega(z)\right)/\left(1 + \alpha z^{q-1}\omega(z)\right), \quad r = |z| \text{ and } 0 < \alpha \leq 1.$$

Proof: An application of (5.1) gives (5.2) easily.

Remark: The transformation $p(z) = (1-\alpha z^{q-1}\omega(z))/(1+\alpha z^{q-1}\omega(z))$ maps the circle $|\omega(z)| \leq r$ onto the circle

$$\left| p(z) - \frac{1+\alpha^2 r^{2q}}{1-\alpha^2 r^{2q}} \right| \leq \frac{2\alpha r^q}{1-\alpha^2 r^{2q}} . \quad \ldots(5.3)$$

Theorem 4—Let $f(z) \in S_q(\alpha)$ and let $\alpha_0 \equiv \{(q+1) - \sqrt{(q+1)^2 - 1}\}$ $\times \left\{\frac{q+2}{\sqrt{(q+1)^2-1}}\right\}$ be the smallest positive root of the equation

$$\{(q+1)^2 - 1\}\alpha^{2/q} - 2\{(q+1)^2-1\}^{1/2}\{(q+1) - \sqrt{(q+1)^2-1}\}^{1/q}\alpha^{1/q}$$
$$- \{(q+1)^2-1\}\{(q+1)-\sqrt{(q+1)^2-1}\}^{2/q} = 0.$$

Then

(i) for $\alpha_0 \leq \alpha \leq 1$, $f(z)$ is convex in

$$|z| < r_1 \equiv \left\{\frac{(q+1) - \sqrt{(q+1)^2-1}}{\alpha}\right\}^{1/q},$$

(ii) for $0 < \alpha \leq \alpha_0$, $f(z)$ is convex in

$$|z| < r_2,$$

where r_2 is the smallest positive root of the equation

$$2\alpha^2 r^{2q} - 2\alpha^2 r^{2q+2} + \alpha q(2+q) r^{q+3} + 2\alpha\{2 - q(q+2)\} r^{q+1}$$
$$+ \alpha q(q+2) r^{q-1} + 2r^2 - 2 = 0.$$

The bounds in (i) and (ii) are both sharp.

PROOF: Since $f(z) \in S_q(\alpha)$, by (2.1), we have

$$z\frac{f'(z)}{f(z)} = \frac{1 - z^q \psi(z)}{1 + z^q \psi(z)}$$

where $\psi(z) \in B_\alpha$. Thus we can write $z\psi(z) = \alpha\,\omega(z)$, where $\omega(z) \in D$. Consequently

$$z\frac{f'(z)}{f(z)} = \frac{1 - \alpha z^{q-1}\omega(z)}{1 + \alpha z^{q-1}\omega(z)} . \quad \ldots(5.4)$$

Differentiating (5.4) logarithmically we have

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1 - \alpha z^{q-1}\omega(z)}{1 + \alpha z^{q-1}\omega(z)}$$
$$- 2\alpha\left\{\frac{z^q\omega'(z) + (q-1)z^{q-1}\omega(z)}{(1+\alpha z^{q-1}\omega(z))(1-\alpha z^{q-1}\omega(z))}\right\}.$$

An application of Lemma 3 to the above equation gives

$$\operatorname{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1}{2\alpha}\left[\operatorname{Re}\left\{\alpha(2+q)\,p(z) - \frac{\alpha q}{p(z)}\right\} - \frac{r^{2q}\,|\,\alpha p(z) + \alpha\,|^2 - |\,1 - p(z)\,|^2}{r^{q-1}(1-r^2)\,|\,p(z)\,|}\right]. \quad \ldots(5.5)$$

By setting $p(z) = a + \xi + i\eta$, $R^2 = (a+\xi)^2 + \eta^2$ where $a = (1 + \alpha^2 r^{2q})/(1 - \alpha^2 r^{2q})$ and denoting the expression on the right-hand side of (5.5) by $E(\xi, \eta)$, we get

$$E(\xi,\eta) = \frac{1}{2\alpha}\left[\alpha(2+q)(a+\xi) - \alpha q(a+\xi)R^{-2} - \frac{1-\alpha^2 r^{2q}}{r^{q-1}(1-r^2)}(d^2 - \xi^2 - \eta^2)R^{-1}\right] \quad \ldots(5.6)$$

where

$$d = (2\alpha r^q)/(1 - \alpha^2 r^{2q}).$$

Differentiating (5.6) partially w.r.t. η, we get

$$\frac{\partial E}{\partial \eta} = \frac{1}{2\alpha}\,\eta R^{-4} F(\xi,\eta) \quad \ldots(5.7)$$

where

$$F(\xi,\eta) = 2\alpha q(a+\xi) + \frac{(d^2 - \xi^2 - \eta^2)(1 - \alpha^2 r^{2q})}{r^{q-1}(1-r^2)}R + 2\frac{1-\alpha^2 r^{2q}}{r^{q-1}(1-r^2)}R^3.$$

It is easily seen that $F(\xi,\eta) > 0$ for all α, $0 < \alpha \leq 1$, $q \geq 1$ and so (5.7) gives that the minimum of $E(\xi,\eta)$ inside the circle $\xi^2 + \eta^2 \leq d^2$ is attained on the diameter $\eta = 0$.

Hence putting $\eta = 0$ in (5.6) we get

$$\begin{aligned}
M(R) \equiv E(\xi, 0) &= \frac{1}{2\alpha}\left\{\alpha(q+2)(a+\xi) - \alpha q(a+\xi)R^{-2} - \frac{1-\alpha^2 r^{2q}}{r^{q-1}(1-r^2)}(d^2 - \xi^2)R^{-1}\right\} \\
&= \frac{1}{2\alpha}\left\{\alpha(q+2)(a+\xi) - \alpha q R^{-1} - \frac{1-\alpha^2 r^{2q}}{r^{q-1}(1-r^2)} \times [d^2 - (R-a)^2]R^{-1}\right\} \\
&= \frac{1}{2\alpha}\left[\left\{\alpha(q+2) + \frac{1-\alpha^2 r^{2q}}{r^{q-1}(1-r^2)}\right\}R + \left\{\frac{1 - \alpha q r^{q-1} + \alpha q r^{q+1} - \alpha^2 r^{2q}}{r^{q-1}(1-r^2)}\right\}R^{-1} - 2a\frac{1-\alpha^2 r^{2q}}{r^{q-1}(1-r^2)}\right]
\end{aligned}$$

where $R = a + \xi$ and $a - d \leq R \leq a + d$. Thus the absolute minimum of $M(R)$ in $(0, \infty)$ is attained at

$$R_0 = \left\{ \frac{1 - \alpha q r^{q-1} + \alpha q r^{q+1} - \alpha^2 r^{2q}}{\alpha (q + 2) r^{q-1} (1 - r^2) + 1 - \alpha^2 r^{2q}} \right\}^{1/2} \quad ...(5.8)$$

and equals

$$M(R_0) = \frac{1}{\alpha r^{q-1} (1 - r^2)} \{[\{\alpha (q + 2) r^{q-1} (1 - r^2) + 1 - \alpha^2 r^{2q}\}$$

$$\times \{1 - \alpha q r^{q-1} + \alpha q r^{q+1} - \alpha^2 r^{2q}\}]^{1/2} - (1 + \alpha^2 r^{2q})\}. \quad ...(5.9)$$

It is easily seen that $R_0 < a + d$, but R_0 is not always greater than $a - d$. In such a case when $R_0 \notin [a - d, a + d]$ the minimum of $M(R)$ on the segment $[a - d, a + d]$ is attained at $R_1 = a - d$ and equals

$$M(R_1) \equiv M(a - d) = \frac{1 - 2\alpha (q + 1) r^q + \alpha^2 r^{2q}}{(1 + \alpha r^q)(1 - \alpha r^q)}. \quad ...(5.10)$$

It follows from what has been said that the bound r' of convexity for the class $S_q(\alpha)$ is determined either from the equation

$$M(R_0) = 0: \quad ...(5.11)$$

or from the equation

$$M(R_1) = 0. \quad ...(5.12)$$

These two equations coincide when $\alpha \in (0, 1]$, which is determined from the equation

$$R_0 = R_1. \quad ...(5.13)$$

Equations (5.11) and (5.12) may be reduced to the following equations:

$$\alpha^2 r^{2q} - 2\alpha (q + 2) r^q + 1 = 0. \quad ...(5.14)$$

$$2\alpha^2 r^{2q} - 2\alpha^2 r^{2q+2} + \alpha q (q + 2) r^{q+3} + 2\alpha \{2 - q (q + 2)\} r^{q-1}$$

$$+ \alpha q (q + 2) r^{q-1} + 2r^2 - 2 = 0. \quad ...(5.15)$$

From (5.14) and (5.15) we get

$$r' = r_1 = \left\{ \frac{(q + 1) - \sqrt{(q + 1)^2 - 1}}{\alpha} \right\}^{1/q} \quad ...(5.16)$$

$$r' = r_2 \quad ...(5.17)$$

where r_2 is the smallest positive root of the equation (5.15). To obtain the value $\alpha = \alpha_0$ which determines the transition from the formula (5.16) to formula (5.17) we must eliminate r from (5.13) and (5.14). We get

$$\{(q + 1)^2 - 1\} \alpha^{2/q} - 2 \{(q + 1)^2 - 1\}^{1/2} \{(q + 1) - \sqrt{(q + 1)^2 - 1}\}^{1/q} \alpha^{1/q}$$

$$- \{(q + 1)^2 - 1\} \{(q + 1) - \sqrt{(q + 1)^2 - 1}\}^{2/q} = 0. \quad ...(5.18)$$

It is evident that the smallest positive root of the equation (5.18) is given by

$$\alpha_0 = \{(q+1) - \sqrt{(q+1)^2 - 1}\} \left\{\frac{q+2}{\sqrt{(q+1)^2-1}}\right\}^q.$$

It is easily seen that it is impossible to use (5.16) when $0 < \alpha \leq \{(q+1) - \sqrt{(q+1)^2 - 1}\}$. Thus we must use formula (5.17) when $0 < \alpha \leq \alpha_0$ and formula (5.16) when $\alpha_0 \leq \alpha \leq 1$. Functions given by

$$f(z) = \frac{z}{(1+\alpha z^q)^{2/q}}$$

and

$$z\frac{f'(z)}{f(z)} = \frac{1-(1-\alpha z^{q-1})\,bz - \alpha z^{q+1}}{1-(1+\alpha z^{q-1})\,bz + \alpha z^{q+1}}$$

where b is determined from

$$\frac{1-(1-\alpha r^{q-1})\,br - \alpha r^{q+1}}{1-(1+\alpha r^{q-1})\,br + \alpha r^{q+1}} = R_0,$$

show that the results obtained in (i) and (ii) are sharp.

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References

Clunie, J., and Keogh, F. R. (1960). On starlike and convex Schlicht functions. *J. Lond. math. Soc.*, **35**, 229–33. *MR* **22**, 1682.

Libera, R. J. (1964). Some radius of convexity problems. *Duke math. J.*, **31**, 143–58. *MR* **28**, 4099.

Mogra, M. L. (1976). On a class of starlike functions in the unit disc I. *J. Indian math. Soc.*, **40**, 159–61.

Padmanabhan, K. S. (1968). On certain classes of starlike functions in the unit disc. *J. Indian math. Soc.* (*N.S.*), **32**, 89–103. *MR* **39**, 2965.

Polya, G., and Szegö, G. (1964). Aufgaben and Lehrsatze aus der analysis, Vol. 1. Springer-Verlag, Berlin.

Robertson, M. S. (1936). On the theory of univalent functions, *Ann. Math.* (2), **37**, 374–408.

Singh, V., and Goel, R. M. (1971). On radii of convexity and starlikeness of some classes of functions. *J. math. Soc. Japan*, **29**, 323–39. *MR* **43**, 7617.