ON A GENERALIZATION OF INVERSE SEMIGROUPS

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In this paper, the notion of a regular semigroup in which each element possesses a unique left or right unit is introduced and a few of its salient features are examined. Such a semigroup is termed as a pre-inverse semigroup since it would be orthodox and a generalization of left and right inverse semigroups. In addition to the various results set forth on these generalized inverse semigroups, a structure theorem is also obtained. The statement of the structure theorem is as follows:

Theorem—Suppose S is pre-inverse with E as its subsemigroup of idempotents which possesses the maximal semilattice decomposition into rectangular bands given by: $E = \cup_{p \epsilon y} E_p$. If we set $S_p = \{x \epsilon S$/for some $x' \epsilon V(x)$ and for all e in E_p, xx', $x'x$, xex', $x'ex$ belong to $E_p\}$, then each S_p is a bisimple maximal subsemigroup of S with E_p as its band of idempotents and every S_p is pre-inverse. Further, S is a union of these mutually disjoint S_p's iff $\boldsymbol{D}$ is a congruence on S and in such a case, each S_p is completely simple.

An alternative characterization of right inverse semigroups is also given and a few results are proved rather easily with this definition. Also, a result on Clifford semigroups has been added to the list.

Introduction

In this paper, we study a class of regular semigroups which possess a special property, namely that every element has a unique left or right unit. These semigroups, for the sake of definiteness are called 'pre-inverse' semigroups. Pre-inverse semigroups turn out to be orthodox and are generalizations of right and left inverse semigroups. Since inverse semigroups form the most important class of semigroups studied so far, we wish to set forth a few results on pre-inverse semigroups which are generalizations of the results on inverse semigroups. It can also be observed that pre-inverse semigroups yield certain results which are unique and cannot be extended to the class of all orthodox semigroups. An expert account of the theory of inverse semigroups may be found in Chapter 7 of the lucid treatise by Clifford and Preston (1967). Various generalizations of inverse semigroups have appeared as research articles during the past decade and a monograph compiling these is not yet written. However, an observation shows that inverse semigroups are the most important class of semigroups which are worth studying and are structures very near to group structures. The present article is an abstract generalization of inverse semigroups. Surprisingly, the generalization attempted in this article has certain unique results and a structure theorem.

The paper is divided into five sections. Section 1 contains the preliminaries required for a detailed study of pre-inverse semigroups. Section 2 contains a few results on right inverse and inverse semigroups which the author has not seen so far in the literature. Section 2 also contains a new characterization of right inverse semigroups which helps us in cutting short the proofs of certain known results. Section 3 is devoted to pre-inverse semigroups, the attempted generalization of inverse semigroups. Section 4 again contains a few results on orthodox semigroups. Section 5 contains the structure theorem for pre-inverse semigroups and thus justifies the introduction of another class of semigroups.

1. Preliminaries

A semigroup S is a set together with an associative binary operation. A semigroup S without an identity element is adjoined with an identity element 1 and is written as S^1. We introduce the various Green's relations on a semigroup S through the following remarks. If $\boldsymbol{a}$ is an element of a semigroup S, then by the $\boldsymbol{L}$-class of $\boldsymbol{a}$ we mean the set of all elements of S which generate the same principal left ideal as generated by $\boldsymbol{a}$ and is denoted by L_a. In other words, an element $\boldsymbol{b}$ of S is in the $\boldsymbol{L}$-class (left class) L_a of $\boldsymbol{a}$ if and only if $S^1a = S^1b$. $\boldsymbol{R}$-classes (right classes) are defined in the dual manner and are denoted by R_a, R_b, etc., *i.e.*, $b \in R_a$ iff $aS^1 = bS^1$. Also, we write $b \in L_a$ as $a\, \boldsymbol{L}\, b$ and $b \in R_a$ as $a\, \boldsymbol{R}\, b$. We observe that $\boldsymbol{L}$ is a right congruence on S while $\boldsymbol{R}$ is a left congruence on S. The smallest equivalence relation containing both $\boldsymbol{L}$ and $\boldsymbol{R}$ is denoted by $\boldsymbol{D}$ and it can be shown that: $\boldsymbol{D} = \boldsymbol{L} \circ \boldsymbol{R} = \boldsymbol{R} \circ \boldsymbol{L}$. Also, a $\boldsymbol{L}$-class and a $\boldsymbol{R}$-class of a semigroup S meet only when both are contained in a particular $\boldsymbol{D}$-class, *i.e.*, $L_a \cap R_b \neq \phi$ implies $L_a \cap R_b \subset D_a$, where ϕ denotes the null set. We define $\boldsymbol{H} = \boldsymbol{L} \cap \boldsymbol{R}$ and clearly $\boldsymbol{H}$ is an equivalence relation. Two elements a, b in S are said to be $\boldsymbol{J}$-equivalent if they generate the same principal ideal, *i.e.*, if $S^1aS^1 = S^1bS^1$. We then write: aJb or $b \in J_a$. The five relations $\boldsymbol{L}, \boldsymbol{R}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{J}$ are due to Green and are called Green's relations in an arbitrary semigroup. We note the significance of these relations in this article later as Remark 2.10. These Green's relations give rise to various classes of semigroups as detailed below. A semigroup S is called left (right) simple if it contains a unique $\boldsymbol{L}$-class ($\boldsymbol{R}$-class); S is called bisimple if it contains a unique $\boldsymbol{D}$-class; and S is called simple if it contains a unique $\boldsymbol{J}$-class. It can also be noted that a semigroup with a unique $\boldsymbol{H}$-class reduces to a group when it contains an idempotent. A simple semigroup S in which there exists an idempotent e with the property: $ef = fe = e \Rightarrow e = f$ for some idempotent f, is called completely simple. Such a semigroup is a rectangular band of groups, where by a *band* we mean a semigroup in which every element is an idempotent. In a rectangular band no two distinct elements commute. These definitions may be extended to semigroups containing zero as detailed in Chapter 2 of the Lucid Treatise by Clifford and Preston (1961). Finally we observe: $\boldsymbol{H} \subset \boldsymbol{L}\, [\boldsymbol{R}] \subset \boldsymbol{D} \subset \boldsymbol{J}$, as the chain of inclusions for the Green's relations and hence any left or right simple semigroup is bisimple and every bisimple semigroup is simple.

We now recall a few classes of semigroups which have drawn wide attention after the work of John Von Neumann in 1936 on regular rings. Following Von

Neumann we define a regular semigroup as one in which to every element x there exists at least one element y such that $xyx = x$. Such an element y is referred to as an inverse of x. If the idempotents of a regular semigroup form a subsemigroup, the semigroup is termed orthodox. Orthodox semigroups in which every element has a unique inverse, or if the idempotents form a semilattice (=commutative band) are called inverse semigroups. Inverse semigroups were introduced by Preston in 1954 and in 1968 Meakin generalized the same to orthodox semigroups. There are a lot more generalizations of inverse semigroups available in the literature and are not required here. Other generalizations of inverse semigroups to left and right inverse semigroups were initiated by Venkateshan (1972) and (independently) by Bailes (1973). We present here a generalization of left and right inverse semigroups tentatively termed as pre-inverse semigroups for the sake of convenience.

A few results from Clifford and Preston (1961) have been repeatedly used. For instance Lemma 1.13, Lemma 2.12, Theorem 2.17 and Theorem 2.18 have been often used. In order to have an immediate reference we merely reproduce without proofs three such results which are frequently used.

Result 1.1—An element a of a semigroup S is regular if and only if the principal right (left) ideal of S generated by a has an idempotent generator.

Result 1.2—If a is a regular element of a semigroup S with a' as one of its inverses, then the elements a, a', aa', $a'a$ are all in the same $\boldsymbol{D}$-class D_a.

Result 1.3—If a is a regular element of a semigroup S, then no $\boldsymbol{H}$-class of S can contain more than one inverse of a.

Notations—For the sake of uniformity we use the following notations in this paper throughout:

S = a semigroup, frequently a regular semigroup;

E = the set of idempotents of the semigroup S, mostly a subsemigroup of S;

ϕ = the null set [in the treatises by Clifford and Preston (1961, 1967) this is denoted by □];

a, b = typical members of S;

e, f = arbitrary members of E, the set of idempotents of S;

$|A|$ = number of elements in the set A, if A is finite and in case A is not finite, this denotes the cardinality of A;

$V(a)$ = the set of all inverses in S of a regular element a (this set is assumed to be empty for non-regular elements);

$aV(a)$ = the set of all elements of the form aa' where a is a regular element of S and a' is an arbitrary inverse of a.

$\boldsymbol{L}, \boldsymbol{R}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{J}$ = Green's relations

L_a, R_a, D_a, H_a, J_a = the classes of Green containing the element a;

Y = an arbitrary semilattice.

2. Right Inverse Semigroups

We recall the following definition and also a result from the papers of Venkateshan (1972) and Bailes (1973). It may be observed that there is a difference in convention between Venkateshan (1972) and Bailes (1973). We use the following convention given by Bailes (1973) throughout this paper.

Definition 2.1 (see Venkateshan 1972, p. 37)—A semigroup S is called a right (left) inverse semigroup iff S is regular and every principal right (left) ideal of S contains a unique idempotent generator. Making use of Lemma 8 of Venkateshan (1972) and Theorem 3 of Bailes (1973) together with the definition of right inverse semigroups given by Bailes (1973), we obtain the following result:

Theorem 2.2—The following statements are equivalent for any regular semigroup S:

(a) S is right inverse;

(b) Every element of S has a unique left unit, *i.e.*, if a is in S then for any two inverses a', a'' of a in S we have: $aa' = aa''$;

(c) Every $\boldsymbol{R}$-class of S contains a unique idempotent;

(d) $Se = Sef = Sfe$ for any two idempotents e, f in S;

(e) $fef = fe$ for any two idempotents e, f in E, the set of idempotents of S.

We characterize right inverse semigroups by a new alternative definition which is not indicated in the papers by Bailes (1973) and Venkateshan (1972).

Definition 2.3—A semigroup S is right inverse iff it is regular and for each a in S, the set $V(a)$ of inverses of a in S is contained in a unique $\boldsymbol{L}$-class of S. To prove the equivalence of the two definitions 2.1 and 2.3 it suffices to prove the following result in view of Theorem 2.2.

Proposition 2.4—Suppose S is regular and $a \in S$. Then $|aV(a)| = 1$ (*i.e.*, a has a unique left unit) iff $a', a'' \in V(a)$ imply $a'\boldsymbol{L}a''$.

Proof: $\Rightarrow$ Let $|aV(a)| = 1$ and $a', a'' \in V(a)$. Then, $aa' = aa''$. Hence $a' = a'aa''$ and $a'' = a''aa'$. These imply: $a'a = (a'a)\ (a''a)$ and $a''a = (a''a)\ (a'a)$. So, $a'a\ \boldsymbol{L}a''a$ or $a'\boldsymbol{L}a''$, since $\boldsymbol{L}$ is a right congruence.

$\Leftarrow$ Conversely, let $a', a'' \in V(a)$ imply $a'\boldsymbol{L}a''$. Hence, $a' = xa''$ for some x in S. So, $a'aa'' = xa''aa'' = xa'' = a'$ or $aa'aa'' = aa'$ or $aa'' = aa'$. Hence the result.

Remark 2.5: Using Definition 2.3 many results proved by Bailes (1973) and Venkateshan (1972) may again be proved in an alternative manner. Some of the proofs are shortened as can be seen by the following Propositions 2.6 and 2.7 proved as Theorem 3 and Theorem 7 (a) by Bailes (1973). The remaining results might also be proved in an alternative manner.

Proposition 2.6—A semigroup S is right inverse iff for each $a \in S$, R has a unique idempotent.

PROOF: $\Rightarrow$ Suppose S is right inverse and $a \in S$. Choose $e, f \in E \cap R_a$. Clearly, since $e, f \in V(e)$, by Definition 2.3 we have $e\boldsymbol{L}f$, since $e\boldsymbol{R}f$ implies $ef = f$ and $fe = e$ which in turn give $efe = e$ and $fef = f$. Therefore, $e\boldsymbol{H}f$ or by using Result 1.3, $e = f$. Thus R_a contains a unique idempotent.

$\Leftarrow$ Conversely, let R_a have a unique idempotent e. Then, $aa' = aa'' = e$ for every $a', a'' \in V(a)$. Therefore, $a' La''$. By Definition 2.3, S is right inverse.

Proposition 2.7—A right inverse semigroup S is orthodox.

PROOF: Let $e \in E$ and $a \in V(e)$. Then eLa and $ae \in E$. Hence, $ae = a \in E$, since e is the right identity of L_a. Hence by Reilly and Scheiblich (1967, Lemma 1.3) it follows that S is orthodox.

Before proceeding further with a few more results on right inverse semigroups, we present a result on inverse semigroups which helps us to characterize several classes of semigroups again by using Green's relations.

Theorem 2.8—The following statements are equivalent for a semigroup S:

(a) S is inverse and is a union of groups;

(b) S is a semilattice of groups;

(c) Every idempotent of S is in the centre;

(d) S is regular and $ab = ba$ whenever ab, ba are idempotents;

(e) S is regular and every $\boldsymbol{J}$-class of S has a unique idempotent;

(f) S is regular and every $\boldsymbol{D}$-class of S has a unique idempotent;

(g) S is regular and $\boldsymbol{H} = \boldsymbol{L} = \boldsymbol{R} = \boldsymbol{D} = \boldsymbol{J}$.

Any commutative regular semigroup possesses the above seven properties.

PROOF: This theorem is due to Clifford who proved it in 1941 and the semigroups satisfying the above equivalent statements are completely regular inverse semigroups (also known as Clifford semigroups). The equivalences of (a), (b), (c), (g) may be found in Chapter 7 of Clifford and Preston (1967) (see results on semilattices of inverse semigroups). We now show that (d) is equivalent to (c). Assuming the validity of (c), by using (a) we find that S is regular. Also, if ab and ba are idempotents, by successively using (c) we obtain: $ab = abab = baab = baba = ba$, which proves ($d$). If we assume ($d$) and consider an $a \in S$, then there exists $x \in S$ such that $axa = a$ or ax, $xa \in E$ or $ax = xa$ and hence x is unique and commutes with a. Since a is arbitrary, by Clifford's result it follows that S is inverse and is a union of groups. Thus, (c) and (d) are equivalent. Clearly, (e) implies (f) since $\boldsymbol{D} \subset \boldsymbol{J}$. Assuming ($f$) we find that S is inverse, since every $\boldsymbol{L}$-class and every $\boldsymbol{R}$-class contain unique idempotents in view of the relation: $\boldsymbol{L}(\boldsymbol{R}) \subset \boldsymbol{D}$. Also, if L is any $\boldsymbol{L}$-class and R is any $\boldsymbol{R}$-class, then R and L must contain the same unique idempotent, say e, since $\boldsymbol{D} = \boldsymbol{R} \circ \boldsymbol{L}$. Therefore, $e \in R \cap L$ or e is in a $\boldsymbol{H}$-class, which must be a group. In other words, all $\boldsymbol{H}$-classes are groups and so S is a union of groups. Therefore, (f) implies (a). We now prove (c) implies (e) thus proving the equivalence of all the statements (a)–(g). Let $\boldsymbol{J}$ be a J-class of S. If $e, f \in J \cap E$ then $S^1eS^1 = S^1fS^1$ and hence there exist x, y, z, u in S such that $e = xfy$ and $f = zeu$. Therefore, $e = f(xy)$ and $f = e(zu)$ or $S^1e = S^1f$ or $e\boldsymbol{L}f$. Hence, $ef = e$ and $fe = f$. But by (c) since all idempotents are in the centre of S, $ef = fe$ or $e = f$, which shows that (e) is valid. This proves the theorem.

Note: The implication, $(a) \Rightarrow (f)$ may also be proved directly as follows: If D is a $\boldsymbol{D}$-class of S and $e, f \in D \cap E$, then there exists an element $a \in S$ such that $e\boldsymbol{R}a$ and $a\boldsymbol{L}f$. In case S is inverse and a union of groups, then $e = aa^{-1} = a^{-1}a = b$ where a^{-1} denotes the unique inverse of $a \in S$. Hence D has a unique idempotent.

Corollary 2.9—A simple inverse semigroup which is also a union of groups is a group.

Remark 2.10: Theorem 2.8 helps us to characterize certain classes of semigroups as detailed below:

(I) A semigroup S is a union of groups if every $\boldsymbol{H}$-class contains a unique idempotent.

(II) A semigroup S is left (right) inverse if every $\boldsymbol{L}$-class ($\boldsymbol{R}$-class) contains a unique idempotent.

(III) A semigroup S is inverse and a union of groups (*i.e.*, a Clifford semigroup) if every $\boldsymbol{D}$-class of S contains a unique idempotent.

(IV) A semigroup S is inverse and a union of groups if every $\boldsymbol{J}$-class contains a unique idempotent.

The present paper deals with a class of semigroups which are generalized versions of class (II) semigroups detailed above. Actually, in the above classification, (III) and (IV) are identical and yield Clifford semigroups while (I) is a generalization of (III) and (II) is a further generalization of (I).

We collect a few more results on right inverse semigroups not available in Bailes (1973) and Venkateshan (1972) through the following remarks. The validity of these statements may be proved easily.

Remark 2.11: (A) In a left cancellative regular semigroup all the idempotents lie in the same $\boldsymbol{R}$-class. Hence, if a left cancellative regular semigroup is right inverse, then it reduces to a group. More generally, a right inverse semigroup is a union of groups iff $V(a) \subset L_a$ for every a in S which in turn is true iff $\boldsymbol{D} = \boldsymbol{L}$. In such a case, $\boldsymbol{L}$ is a congruence on S, S is a band of maximal left groups and $S/\boldsymbol{L}$ is a semilattice of groups.

(B) Suppose S is a regular semigroup with E as its set of idempotents and $c^* = ((e, f) \in E \times E / e\boldsymbol{R}f)$. Let c be the smallest congruence on S containing c^*. Then c is the smallest congruence on S such that S/c is right inverse. In particular if S is an orthodox semigroup and c the smallest congruence on S containing $\boldsymbol{R}$, then c is the smallest congruence on S such that S/c is right inverse.

(C) If S is a right inverse semigroup, then every idempotent-separating congruence on S is contained in $\boldsymbol{H}$. Moreover, the set of all idempotent-separating congruences on S forms a modular sublattice of the lattice of all congruences on S with a greatest and a least element. The greatest element of this sublattice, namely the maximum idempotent-separating congruence $\boldsymbol{i}$ on S is given by: $\boldsymbol{i} = \{(a, b)/ aea' = beb'$ for all e in E and for some inverses a', b' of a, b in $S\}$. (For more details we refer to Howie 1964 and Munn 1964.)

Proposition 2.12—Any right inverse semigroup may be embedded in a simple right inverse semigroup with identity.

PROOF: Suppose S is a right inverse semigroup with an identity 1. (Otherwise, without any loss in generality we can adjoin an identity 1 to S.) Let $C(S)$ be a semigroup generated by S and two elements a, b not in S subject to the relations $ab = 1$, $as = a$, $sb = s$ for all s in S and for such equations which hold in S. Then $C(S)$ is a simple semigroup with identity in which S is embedded [see pp.108–111 of Clifford and Preston (1967) for full details]. The inverse of an element $b^j s' a^i$ in $C(S)$ is of the form $b^i s a^j$ where s' denotes an inverse of s in S. We have: $ss' = ss''$ iff $b^i (ss') a^i = b^i (ss'') a^i$ iff $(b^i s a^j)(b^j s' a^i) = (b^i s a^j)(b^j s'' a^i)$. Therefore, S is right inverse iff $C(S)$ is right inverse. The result follows.

Note: There exist simple right inverse semigroups with identity having an arbitrary number of $\boldsymbol{D}$-classes.

Example—Suppose S denotes the set of nonzero complex numbers and we define a new operation o on S by the rule: $xoy = x\,|\,y\,|$, for every x, y in S. Then S is a regular semigroup under the new operation with various properties—(i) S is a union of groups; (ii) S is right cancellative and hence a left group; (iii) S is right inverse and hence orthodox; (iv) S is completely simple without zero. However, S is not an inverse semigroup since in a completely simple semigroup without zero, distinct idempotents never commute whereas in an inverse semigroup the idempotents commute and form a semilattice which implies that a completely simple inverse semigroup without zero is a group. Actually, every element of S possesses an infinite number of inverses. Geometrically, we can visualize S as the union of mutually disjoint groups, each group being a ray through the origin of course, with the origin deleted. Therefore we conclude that a right cancellative right inverse semigroup need not be a group. It is, however, clear that a left cancellative right inverse semigroup is a group.

3. PRE-INVERSE SEMIGROUPS

The attempted generalization of right and left inverse semigroups are called pre-inverse semigroups for the sake of convenience. These semigroups are all orthodox and therefore our attention would be confined to only a subclass of the class of all orthodox semigroups. The formal definition of a pre-inverse semigroup is given below:

Definition 3.1—Suppose S is a regular semigroup. Then S is called 'pre-inverse' if every element of S has a unique left or right unit.

Note: Regularity assumption is not necessary but may be proved as a consequence. However, we assume that S is regular without any loss in generality.

Examples—A left (right) zero semigroup is right (left) inverse and more generally, a left (right) group is a right (left) inverse semigroup. Any left or right inverse semigroup is pre-inverse. Nontrivially, a 0-disjoint union of a right and a left inverse semigroup yields a pre-inverse semigroup. In other words, let A (B) be a right (left) inverse semigroup not containing 0 and set: $S = A \cup B \cup \{0\}$. Define (.) on S by:

$$x \, . \, y = \begin{cases} xy \text{ if } x \text{ and } y \text{ both belong to } A \text{ or both belong to } B. \\ 0, \text{ otherwise.} \end{cases}$$

Then $(S, .)$ is a pre-inverse semigroup.

Note: We recall that if $a \in S$ is a regular element, then $V(a)$ denotes the set of all inverses of a. By $aV(a)$ we mean the set of all elements of the form aa-where $a' \in V(a)$. Also, $|\, aV(a) \,| = 1$ means that a possesses a unique left unit.

Proposition 3.2—A semigroup S is pre-inverse iff each $\boldsymbol{R}$-class R_a whenever $|\, aV(a) \,| = 1$ or each $\boldsymbol{L}$-class L_a whenever $|\, V(a)\, a \,| = 1$ contains a unique idempotent, a being an arbitrary element of S.

PROOF $\Rightarrow$ Suppose S is pre-inverse and $|\, aV(a) \,| = 1$. Since S is regular, by Result 1.1, for any a in S, both R_a and L_a contain idempotents. Let us choose dempotents e, f in $E \subset S$ such that $a \in R_e$ and $e\boldsymbol{R}f$. Let $g \in L_a \cap E$. Then $e \in R_{ge} \cap L_{ge}$ and $g \in L_a \cap R_{ge}$. Hence $R_g \cap L_e$ contains an inverse a' of a. So, $aa' \in R_a \cap L_{a'} = R_e \cap L_e = H_e$. Therefore by Result 1.3, $aa' = e$. Likewise there exists an inverse a'' of a in $R_g \cap L_f$ such that $aa'' = f$. But we have assumed $|\, aV(a) \,| = 1$ or $aa' = aa''$. Therefore, $e = f$ and R_a has a unique idempotent. (A variation of this proof is already given in the first half of the proof of Proposition 2.6.)

$\Leftarrow$ Conversely, for some a in S, let L_a contain a unique idempotent say e. So, $L_a = L_e$ or there exist b, c in S such that $a = be$ and $e = ca$. Hence, $a = bca = bcaca = aca$ and so a is regular. If $a', a'' \in V(a)$, we put $a'a = g$ and $a''a = f$. Then gLa and fLa. Therefore by assumption, $g = f = e$ or $|\, V(a) \,|a = 1$. Similarly, if R_a contains a unique idempotent, we can show that a has a unique left unit. Therefore, S is pre-inverse.

Proposition 3.3—A pre-inverse semigroup is orthodox.

PROOF: Let S be a pre-inverse semigroup with E as its set of idempotents. Let $e \in E, |\, eV(e) \,| = 1$ and $x \in V(e)$. Hence, ex, $xe \in E \cap V(e)$. By Proposition 2.4 $xe\boldsymbol{L}\, ex\boldsymbol{L}e$ and so $xe = x$ since e is a right identity of L_e. Therefore, by Lemma 1.3 of Reilly and Scheiblich (1967), it follows that S is orthodox.

Proposition 3.4—A regular subsemigroup T of a pre-inverse semigroup S is Pre-inverse.

PROOF: By result 9 (ii) of Hall (1970) e, f are idempotents in T and eRf (eLf) in T imply that $e, f \in E \subset S$ and eRf (eLf) in S. Hence the result.

Remark 3.5: If S is pre-inverse, it is clear that E, the set of idempotents of S is a band. Hence, by using the result of McLean we can express E as a union of disjoint rectangular bands. That is $E = \cup_{p \in Y} E_p$ where Y is a semilattice, each E_p is a rectangular band and $E_p \cap E_q = \phi$ for $p \neq q$, with $E_p E_q \subset E_{pq}$. (This result is true for orthodox semigroups).

Remark 3.6: Using Definition 2.3 we can characterize a pre-inverse semigroup as a regular semigroup in which for any element a, $V(a)$ is contained in a unique $\boldsymbol{L}$-class or a unique $\boldsymbol{R}$-class.

4. Orthodox Semigroups

In this section we set forth a few results on orthodox semigroups. The notations used in Remark 3.5 for splitting the band of idempotents of an orthodox semigroup would be made use of below.

Proposition 4.1—If S is orthodox with $E = \cup_{p \in Y} E_p$, $e, f \in E$ with $|eV(e)| = 1$, then $e\boldsymbol{L}f$ iff $e, f \in E_p$ for some p and if $e, f \in E$ with $|V(e)e| = 1$, then $e\boldsymbol{R}f$ iff e. $f \in E_p$ for some p. In particular if S is inverse, then $Y = E$ and each E_p is a group of order one.

PROOF: $\Rightarrow$ If $e, f \in E$ with $|eV(e)| = 1$ and eLf, then $ef = e$ and $fe = f$, Therefore $efe = e$ and $fef = f$ or $f \in V(e) = E_p$ for some p. Hence: $e, f \in E_p$.

$\Leftarrow$ If $e, f \in E$ with $|eV(e)| = 1$ and $e, f \in E_p = V(e)$ then by Proposition 2.4 eLf. The result follows.

Proposition 4.2—Suppose S is orthodox, $a \in S$ and $|aV(a)| = 1$. $(|V(a)a| = 1)$. Let $e \in L_a \cap E$ $(e \in R_a \cap E)$. Then $|V(a)| = |E_p|$ where $e \in E_p$ and $E = \cup_{p \in Y} E_p$.

PROOF: Let us for the sake of definiteness assume that $|aV(a)| = 1$. By Result 1.3, $a\boldsymbol{H}$-class of S contains atmost one inverse of any element, and hence: $|V(a)| = |K|$ where K is the set of all those $\boldsymbol{H}$-classes of S which meet $V(a)$. It is clear that: $V(a) \cap H_b \neq \phi$ iff $R_a \cap L_b$ and $R_b \cap L_a$ are groups. If f is the unique idempotent of R_b, then $f \in L_a \cap R_b \cap E$. Therefore, $H_b \in K$ iff $f \in L_b$. Assuming, fLb, $H_b \in K$ iff $R_b \cap L_a$ contains an idempotent. That is, iff gRb for some $g \in E_p \subset E$. Therefore, $g \leftrightarrow b$ is a one-to-one correspondence of E_p onto H_b. So, $|E_p| = |K|$ and the result follows.

Proposition 4.3—Suppose S is regular and $e \in E$. Then

(i) $(x \in S / x \in V(y)$ for some $y \in R_e$ with $|yV(y)| = 1) \subset L_e \subset (x \in S / x \in V(y)$ for some $y \in R_e)$. Equalities hold whenever R_e has a unique idempotent.

(ii) $(x \in S / x \in V(y)$ or some $y \in L_e$ with $|V(y)y| = 1) \subset R_e \subset (x \in S / x \in V(y)$ for some $y \in L_e)$. Equalities hold whenever L_e has a unique idempotent.

Proof: Similar to Lemma 12 of Bailes (1973, p. 496) and hence is omitted.

Proposition 4.4—Suppose S is orthodox, $e \in E$ and $|eV(e)| = 1$. Then R_e is a subsemigroup of S iff e is a right identity of D_e. In this case R_e is right cancellative with identity.

Proof: For a proof we modify the proof of Theorem 13 of Bailes (1973, p. 496).

Proposition 4.5—Suppose S is orthodox. Then a D-class D of S is a subsemigroup iff the set of idempotents of D is a subsemigroup of E. Further, in this case, D is bisimple orthodox and if S is pre-inverse, so would be D.

Proof: The first part follows from Theorem 14 of Bailes (1973, p. 496). The fact that D is bisimple follows from Exercise 6 of Clifford and Pieston (1961, p. 62). It is clear that D is orthodox, and in case S is pre-inverse, by Proposition 3.4, we conclude that D is pre-inverse.

Proposition 4.6—Suppose S is regular and we define:

$C^* = ((e,f) \in E \times E / e\mathbf{R}f$ or $e\mathbf{L}f)$. Let C be the smallest congruence on S containing C^*. Then S/C is a pre-inverse semigroup.

Proof: Let e^*, f^* be idempotents in S/C such that $e^*\mathbf{R}f^*$ in S/C. That is $e^*f^* = f^*$ and $f^*e^* = e^*$. Then there exist idempotents e, f, g, h in S such that $eC = e^*$, $fC = f^*$, $g \in V(ef)$, $h \in V(fe)$, $ge = fg = g$ and $hf = eh = h$. Thus $gef \in E$ and $gef = gf\mathbf{R}g$. So, $(gf)C = gC$. Likewise, $(he)C = hC$. It can now be shown that : $(ef)C = gC$ and $(fe)C = hC$. Hence, $gC = hC$ and therefore, $e^* = f^*$. Thus S/C is pre-inverse. (For more details we refer to the proof of Theorem 17 of Bailes 1973, p. 497).

5. Structure Theorem

In this concluding section, our aim is to prove the result of this paper, namely the semilattice decomposition of a pre-inverse semigroup which is a union of groups into maximal completely simple pre-inverse semigroups.

Proposition 5.1—Let S be pre-inverse and be a union of disjoint regular semigroups S_z, $z \in X$, where X is a semilattice. If E_z denotes the set of idempotents S_z, $z \in X$ then S is a semilattice of S_z, $z \in X$ iff E is a semilattice of E_z, $z \in X$ and $a \in S_z \Rightarrow V(a) \subset S_z$.

Proof: If S is a semilattice of S_z, $z \in X$ it is clear that E is a semilattice of E_z, $z \in X$. Let $a \in S_p$ and $a' \in V(a)$ with say $a' \in S_q$ for some $p, q \in X$. Hence, $a = aa'a$ in S_{pq} and a' is also in S_{pq}, since X is a semilattice. Therefore, $p = pq = q$ and so, $V(a) \subset S_p$.

Conversely, if E is a semilattice of E_z, $z \in X$ and $a \in S_z \Rightarrow V(a) \subset S_z$, choose $a \in S_p$, $b \in S_q$, $a' \in V(a)$, $b' \in V(b)$ for some $p, q \in X$. Suppose $abb' \in S_r$ for some r in X. Since S_r is pre-inverse it is orthodox by Proposition 3.3. Hence, $abb'a' = (abb')(bb'a') \in E_r$ since $bb'a' \in V(abb')$. Also, $(bb'a')(abb') \in E_r$. But, $(bb'a')$

$(abb') = (bb')\ (a'a)\ (bb') \in E_{pq}$. Therefore, $r = pq$. So if, $ab \in S_t$ for some $t \in X_z$ then $abb'a' \in S_t$ since $b'a' \in V(ab)$. But, $abb'a' \in E_r = E_{pq}$. Therefore, $t = p_z$ or $S_p S_q \subset S_{pq}$. Hence S is a semilattice of S_z, $z \in X$.

Proposition 5.2—A semilattice of pre-inverse semigroups is pre-inverse.

PROOF: Let S be the semilattice of pre-inverse semigroups S_z, $z \in X$. Clearly, S is regular. Choose $e, f \in E$ such that $e\boldsymbol{R}f$ in S, *i.e.*, $ef = e$ and $fe = f$. If $e \in S_z$ and $f \in S_{z'}$, for some $z, z' \in X$ then $e = fe \ \varepsilon\ S_{zz'}$, and $f = ef \ \varepsilon\ S_{zz'}$. Therefore, $z = zz' = z'$ or $e, f \ \varepsilon\ S_z$.

Since S_z is pre-inverse and so a regular subsemigroup of S, it follows that $e\boldsymbol{R}f$ in S_z. Likewise $e\boldsymbol{L}f$ in S implies $e\boldsymbol{L}f$ in S_z. But S_z is pre-inverse. Hence, one of $e\boldsymbol{R}f$ or $e\boldsymbol{L}f$ implies that $e = f$. Therefore, S is pre-inverse.

From Lemma 22 and Corollary 23, of Bailes (1973, p. 500) we obtain the following:

Proposition 5.3—If S is orthodox, $a \in S$, $|\, aV(a)\, | = 1$, then H_a is a group iff $a\boldsymbol{L}a'$ for some $a' \in V(a)$. In this case $a\boldsymbol{L}a^*$ for all a^* in $V(a)$.

(Another proof of Proposition 5.3 may be obtained by using Definition 2.3. For the sake of brevity we omit the proof.)

Remark 5.4—Suppose S is orthodox with E as its band of idempotents and let E be the maximal semilattice decomposition into rectangular bands E_p, $p \in Y$, a semilattice. Hence $\boldsymbol{D}$ is a congruence on E by Theorem 1.5 of Krishna Iyengar (1971). For each $p \in Y$, we define:

$S_p = (x \in S/\text{for some } x' \in V(x) \text{ and for all } e \in E_p,\ xx',\ x'x,\ xex',\ x'ex \in E_p)$. Then by Theorem 1.5 of Reilly and Scheiblich (1967) it is clear that each S_p is the maximal orthodox subsemigroup of S with E_p as its band of idempotents. We make use of the same notations as used in this remark in what follows.

Proposition 5.5—If S is pre-inverse then S_p is the maximal pre-inverse subsemigroup of S with E_p as its band of idempotents. Further, each S_p is bisimple and $S_p \cap S_q = \phi$ if $p \neq q$.

PROOF: By proposition 3.4 since S_p is a regular subsemigroup of S, it follows that S_p is pre-inverse. Let $x \in S_p \cap S_q$ and $|\, xV(x)\, | = 1$. Hence there exist x', $x'' \in V(x)$ such that $xx' \in E_p$ and $xx'' \in E_q$. Since $|\, xV(x)\, | = 1$, $xx' = xx'' \in E_p \cap E_q$. So, $p = q$ and the sets S_p, $p \in Y$ are disjoint. Since S_p is regular, every $\boldsymbol{D}$-class of S_p contains an idempotent. Since S is pre-inverse, $e\boldsymbol{L}f$ iff $e\ f \in E_p$ ($e\boldsymbol{R}f$ iff e, f $e, f \in E_p$) whenever $|\, eV(e)\, | = 1$ $(|\, V(e)\, e\, | = 1)$ by Proposition 4.1. Since S_p is pre-inverse with E_p as its band of idempotents, it follows that any two elements of E_p are either $\boldsymbol{L}$-related or are $\boldsymbol{R}$-related or both. Hence, $e, f \in E_p$ forces $e\boldsymbol{D}f$. Therefore, S_p is bisimple.

Corollary 5.6—$S_x = D_x$ for any x in S.

Remark 5.7: Suppose $S = \cup_{p \in Y} S_p = \cup_{p \in Y} (D_x/x \in S_p)$. Hence S is a semilattice of bisimple semigroups. This is because, $S_p = D_x$ for any s in S_p by Corollary 5.6. Hence $V(x) \subset S_p$ by Result 1.2. Therefore, $x \in S_p$ implies that $V(x) \subset S_p$ and clearly E is a semilattice of E_p, $p \in Y$. So by Proposition 5.1, S is semilattice of S_p, $p \in Y$. Now the equivalent statements (A), (B), (C), (E) of Theorem 1.5 of Krishna Iyengar (1971) show that S is band of S_p in which $\boldsymbol{D}$ is a congruence and $ab\ \boldsymbol{D}\ ba$ for all a, b in S. Further, each S_p is completely simple, because $ef = fe = e$ in S_p implies $e, f \in E_p$ and $ef = fe$ implies $e = f$ since E_p is a rectangular band and so nowhere commutative. Therefore S_p is a band of groups by Corollary 2.52 (*b*) of Clifford and Preston (1961). Hence S is a union of groups in such a case. Conversely, let S be a union of groups and $x \in S$. Then $V(x) \subset D$. by Result 1.2 and E is a semilattice of E_p, $p \in Y$ would imply that there is a unique $p \in Y$ such that $E_p \cap D_x \neq \phi$ and hence $E_p \subset D_p$. Now by Result 1.2 $x'x\ xx' \in E_p$ for all $x' \in V(x)$. Clearly every $\boldsymbol{H}$-class of S is a group by Theorem 4.3 of Clifford and Preston (1961). If $e \in E_p$, then $xex' \in R_{xe} \cap L_{x'} \subset D_x$ and $x'ex \in R_{x'} \cap L_{ex} \subset D_x$. But xex', $x'ex$ are idempotents by Lemma 1.4 of Reilly and Scheiblich (1967). Therefore, xex', $x'ex \in E_p$. So by definition, $x \in S_p$. Hence $x \in S$ implies that $x \in S_p$ for a unique $p \in Y$. Summarizing the above remarks, we obtain the following:

Structure Theorem—Suppose S is pre-inverse, $E = \cup_{p \in Y} E_p$ the maximal semilattice decomposition of E into rectangular bands and $S_p = (x \in S/\text{for some } x' \in V(x)$ and for all $e \in E_p$, $xx', x'x, xex', x'ex \in E_p)$. Then each S_p is a bisimple maximal pre-inverse subsemigroup of S with E_p as its band of idempotents and $S_p \cap S_q = \phi$ if $p \neq q$. Further, the following conditions are equivalent in such a case:

(1) $S = \cap_{p \in Y} S_p$;

(2) S is a semilattice of S_p, $p \in Y$;

(3) For every $p \in Y$, $S_p = D_x$ for all x in S_p;

(4) S is a semilattice of bisimple semigroups;

(5) S is a band of bisimple semigroups;

(6) $\boldsymbol{D}$ is a congruence on S;

(7) $ab\ \boldsymbol{D}\ ba$ for all a, b in S;

(8) S is a union of groups.

Moreover, in such a case, each S_p is completely simple and hence a rectangular band of groups.

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References

Bailes, G. L. (Jr.) (1973). Right inverse semigroups. *J. Alg.*, **26**, 492–507.

Clifford, A. H., and Preston, G. B. (1961). The Algebraic Theory of Semigroups, Vol. I. Mathematical Surveys, No. 7, *AMS*.

——— (1967). The Algebraic Theory of Semigroups, Vol. II. Mathematical Surveys, No. 7, *AMS*.

Hall, T. E. (1970). On orthodox semigroups and uniform and antiuniform bands. *J. Alg.*, **16**, 204–17.

Howie, J. M. (1964). The maximum idempotent-separating congruence on an inverse semigroup. *Proc. Edin. math. Soc.*, **14**, 71–79.

Krishna Iyengar, H. R. (1964). Semilattices of bisimple regular semigroups. *Proc. AMS*, **28**, 361–65.

Munn, W. D. (1964). A certain sublattice of the lattice of all congruences on a regular semigroup. *Proc. Camb. phil. Soc.*, **60**, 385–91.

Reilly, N. R., and Scheiblich, H. E. (1967). Congruences on regular semigroups. *Pacific J. Math.*, **23**, 349–60.

Venkateshan, P. S. (1972). Bisimple left inverse semigroups. *Semigroup Forum*, **4**, 34–45.