ON SIMILARITY SOLUTION OF CHEMICAL BOUNDARY LAYER THEORY

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This paper presents an investigation of similarity solutions in chemical boundary layer theory. It is seen that when the flow is accompanied with nth order homogeneous volume reaction, we may get similarity solution when $n \neq 1$.

INTRODUCTION

There is a possibility of an interesting and important investigation in boundary layer theory when the flow is along a soluble body and accompanied by chemical reactions. This type of study is useful in chemical physics and engineering. Some research work in this direction has been done by Ghosal *et al.* (1972) and others.

ANALYSIS

Let us consider the flow along a flat plate that contains a species A slightly soluble in the fluid B. The concentration at the plate surface is assumed to be $\bar{C}_{Aw}$ which is the solubility of A in B and the concentration far from the plate is assumed to be $\bar{C}_{A\infty} = 0$.

We are considering here the case when A reacts with B by an nth order homogeneous chemical reaction (volume). We further assume here that the concentration of dissolved A is small enough to consider the physical properties ρ, μ D_{AB} to be constants throughout the fluid, where ρ and μ are respectively the density and the viscosity of the fluid B and D_{AB} is the diffusivity of A in B.

The boundary layer equations of the above system are adequately described by the following equations (Bird *et al.* 1960):

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \quad \ldots(1)$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad \ldots(2)$$

$$\frac{\partial \bar{C}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{C}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{C}}{\partial \bar{y}} = D_{AB}\frac{\partial^2 \bar{C}}{\partial \bar{y}^2} + K\bar{C}^n \quad \ldots(3)$$

where $\bar{C}$ is the concentration of the solute which is a function of $\bar{x}, \bar{y}, \bar{t}$; n is an integer which indicates the order of the reaction and K the volume rate constant.

The boundary conditions of the system are given by

$$\left.\begin{array}{lll} \bar{y}=0; & \bar{u}=\bar{v}=0, & \bar{C}=\bar{C}_w(\bar{x},\bar{t}) \\ \bar{y}\to\infty; & \bar{u}\to\bar{U}_\infty\,(\text{constant}), & \bar{C}\to 0 \end{array}\right\} \quad \ldots(4)$$

(subscript w is the wall condition).

The variables u, v, x, y, t are made dimensionless as follows:

$$\left.\begin{array}{lll} x=\dfrac{\bar{x}}{L}, & y=\dfrac{\bar{y}}{L}, & t=\dfrac{\nu\bar{t}}{L^2} \\ u=\dfrac{\bar{u}L}{\nu}, & v=\dfrac{\bar{v}L}{\nu}, & C=\dfrac{\bar{C}}{C_0} \end{array}\right\} \quad \ldots(5)$$

where L is a representative length and C_0 denotes a typical value of $\bar{C}$, say at $x=0$ on the plate at $t=0$. Introducing (5) in eqns. (1), (2) and (3) we get

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2} \quad \ldots(6)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \ldots(7)$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{1}{Sc}\frac{\partial^2 C}{\partial y^2}+\frac{C_0^{n-1}L^2K}{\nu}C^n \quad \ldots(8)$$

where

$$Sc=\frac{\nu}{D_{AB}}$$

is the Schmidt's number and

$$\frac{C_0^{n-1}L^2\mathrm{K}}{\nu}$$

is a dimensionless constant.

The boundary conditions now become

$$\left.\begin{array}{lll} y=0; & u=v=0, & C=C_w(x,t) \\ y\to\infty; & u\to U_\infty\,(\text{constant}), & C\to 0 \end{array}\right\} \quad \ldots(9)$$

The equation of continuity is satisfied if we choose a dimensionless stream function ψ such that

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}.$$

If we choose, a similarity variable in the form

$$\eta = y\, \phi_1 (x, t) \qquad \ldots(10)$$

and the dependent variables as follows:

$$f(\eta) = \psi (x, y, t)/\phi_2 (x, t) \qquad \ldots(11)$$

$$\theta(\eta) = AC\ (x, y, t)/C_w\ (x, t) \qquad \ldots(12)$$

(where A is a constant which is to be determined), the present problem reduces to one of determining the functions $\phi_1 (x, t)$, $\phi_2 (x, t)$ and $C_w\ (x, t)$ such that eqns. (6) and (8) are reducible to ordinary differential equations for $f(\eta)$ and $\theta(\eta)$ with proper boundary conditions. With the following identities from eqns. (10), (11) and (12),

$$u = \frac{\partial \psi}{\partial y} = \phi_1 \phi_2 f'(\eta) \qquad \ldots(13)$$

$$v = -\frac{\partial \psi}{\partial x} = -\left[\frac{\partial \phi_2}{\partial x} f + \frac{\phi_2}{\phi_1} \frac{\partial \phi_1}{\partial x} \eta f'\right] \qquad \ldots(14)$$

$$C = \frac{C_w}{A}\ \theta \qquad \ldots(15)$$

substituted in eqns. (6) and (8) we obtain

$$f''' - a_2 \eta f'' - (a_2 + a_3) f' + a_5 ff'' - (a_4 + a_5) f'^2 = 0 \qquad \ldots(16)$$

$$\frac{1}{Sc}\ \theta'' - (a_2 \eta - a_5 f)\, \theta' - (a_6 + a_7 f')\, \theta + a_1\, \theta^n = 0 \qquad \ldots(17)$$

where

$$a_1 = C_w^{n-1}/\phi_1^2 \qquad \ldots(18)$$

$$a_2 = \phi_{1t}/\phi_1^3 \qquad \ldots(19)$$

$$a_3 = \phi_{2t}/(\phi_1^2/\phi_2) \qquad \ldots(20)$$

$$a_4 = (\phi_2 \phi_{1x})/\phi_1^2 \qquad \ldots(21)$$

$$a_5 = \phi_{2x}/\phi_1 \qquad \ldots(22)$$

$$a_6 = C_{wt}/(C_w\, \phi_1^2) \qquad \ldots(23)$$

$$a_7 = \phi_2\, C_{wx}/(\phi_1 C_w) \qquad \ldots(24)$$

It is to be noted here that A is made equal to $C_0 (L^2 k/\nu)^{1/(n-1)}$. We now try to find out for what values of n similarity solutions for eqns. (16) and (17) are possible. From eqn. (19) we get

$$\phi_1 = [B(x) - 2a_2 t]^{-\frac{1}{2}} \qquad \ldots(25)$$

where $B(x)$ is yet to be determined. Combining eqns. (19) and (20) we get

$$\phi_2 = D(x)\, \phi_1^{m} \qquad \ldots(26)$$

where $D(x)$ is an arbitrary function and $w = a_3/a_2$.

Thus

$$\phi_2 = D(x)\,[B(x) - 2a_2 t]^{-w/2}. \qquad ...(27)$$

A similar combination of eqns. (21) and (22) gives

$$\phi_2 = E(t)\,\phi_1^{\varepsilon} \qquad ...(28)$$

where $\varepsilon = a_5/a_4$, $a_4 \neq 0$. The case of $a_4 = 0$ will be treated later. Equation (21) may be solved giving

$$\phi_1 = \left[\frac{a_4(\varepsilon - 1)}{E(t)}\, x + F(t)\right]^{1/(\varepsilon - 1)} \qquad ...(29)$$

and therefore ϕ_2 is given by

$$\phi_2 = E(t)\left[\frac{a_4(\varepsilon - 1)}{E(t)}\, x + F(t)\right]^{\varepsilon/(\varepsilon - 1)} \qquad ...(30)$$

Comparing eqns. (25), (27), (29) and (30), we get

$$\varepsilon = w = -1, \qquad B(x) = -\frac{2a_4}{a_{12}}\, x$$

$$D(x) = E(t) = a_{12}, \qquad F(t) = -2a_2 t.$$

With the above results we get

$$\phi_1 = \left(-\frac{2a_4}{a_{12}}\, x - 2a_2 t\right)^{-1/2} \qquad ...(31)$$

$$\phi_2 = a_{12}\left(-\frac{2a_4}{a_{12}}\, x - 2a_2 t\right)^{1/2} \qquad ...(32)$$

From relations (18) and (31) it appears that

$$C_w = a_1^{1/(n-1)}\left(-\frac{2a_4}{a_{12}}\, x - 2a_2 t\right)^{-1/(n-1)}$$

$$= a_{13}\left(-\frac{2a_4}{a_{12}}\, x - 2a_2 t\right)^{-1/(n-1)} \qquad ...(33)$$

where $a_{13} = a_1^{1/(n-1)}$.

Now it is seen that the right-hand sides of relations (18)–(24) become constant with the above results for ϕ_1, ϕ_2 and C_w. We see that n may be any value except 1. Thus similarity solutions of eqns. (16) and (17) are possible if $n \neq 1$.

The case of $a_4 = 0$ will be considered now. If $a_4 = 0$ then eqn. (25) becomes

$$\phi_1 = [a_{14} - 2a_2 t]^{-\frac{1}{2}} \qquad ...(34)$$

where a_{14} is a constant of integration.

From eqns. (20), (22) and (34) we ultimately get

$$\phi_2 = (a_5 x + a_{15})(a_{14} - 2a_2 t)^{-\frac{1}{2}} \quad \ldots(35)$$

where a_{15} is a constant.

Also from eqns. (18) and (34) we get

$$C_w = b\,(a_{14} - 2a_2 t)^{-1/(n-1)}$$

where $b = a_1^{1/(n-1)}$.

It can be easily verified that the right-hand sides of relations (23) and (24) are constants with the above results for ϕ_1, ϕ_2 and C_w. Hence similarity solution is also possible in this case for any value of n except $n = 1$.

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