PARABOLIC BOUNDARY VALUE PROBLEM

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In the present paper a linear one-dimensional parabolic boundary value problem has been solved by employing semi-numerical technique. The solution is obtained in the form of an infinite series.

1. INTRODUCTION

A linear one-dimensional parabolic boundary value problem for the time interval $(0, T_1 + T_2)$ is considered here; for applications of its particular cases to physical situations please refer to Reed and Mullineux (1973), Wankhede (1975), and Howard and Sutton (1970 *a*, *b*). The solution is obtained with the help of a semi-numerical procedure.

2. STATEMENT OF THE PROBLEM

In the interval $(0, T_1)$, the function $V(x, t)$ satisfies

$$\frac{1}{p(x)} \Lambda_x [V(x, t)] = \frac{1}{\nu} \frac{\partial V}{\partial t}(x, t)$$

$$a < x < b, \quad 0 < t < T_1, \qquad \ldots(2.1)$$

with

$$V(x, t) = G(x) \quad \text{at } t = 0, \quad a < x < b, \qquad \ldots(2.2)$$

$$V(x, t) = H(x) \quad \text{at } t = T_1, \quad a < x < b, \qquad \ldots(2.3)$$

$$N_a [V] = A_1, \quad 0 < t < T_1, \qquad \ldots(2.4)$$

and

$$N_b [V] = A_2, \quad 0 < t < T_1, \qquad \ldots(2.5)$$

where

$$\Lambda_x = \frac{\partial}{\partial x}\left[r(x) \frac{\partial}{\partial x}\right] - q(x),$$

in which the functions $p(x)$, $q(x)$, $r(x)$ and $dr/(x)\,dx$ are real-valued and continuous, $p(x) > 0$ and $r(x) > 0$, over the entire interval $a \leq x \leq b$,

$$N_a [F] = \left[a_1 F(x, t) + a_2 \frac{\partial F}{\partial x}(x, t)\right]_{x=a},$$

and $H(x)$ and $G(x)$ are the unknown functions whereas $\nu > 0$, A_1 and A_2 are real constants.

And, in the interval $(T_1, T_1 + T_2)$, the function $V(x, t)$ satisfies

$$\frac{1}{p(x)} \Lambda_s [V(x, t)] = \frac{1}{v} \frac{\partial V}{\partial J}(x, t)$$

$$a < x < b, \qquad 0 < J < T_2, \qquad \ldots(2.6)$$

with

$$V(x, J) = H(x) \quad \text{at } J = 0, a < x < b, \qquad \ldots(2.7)$$

$$V(x, J) = G(x) \quad \text{at } J = T_2, \quad a < x < b, \qquad \ldots(2.8)$$

$$N_a[V] = A_3, \quad 0 < J < T_2, \qquad \ldots(2.9)$$

and

$$N_b[V] = A_4, 0 < J < T_2, \qquad \ldots(2.10)$$

$J = t - T_1$ and A_3, A_4 are all real constants.

3. Solution of the Problem

First consider the set of equations (2.1) to (2.5). Applying the generalized Fourier transform defined by Churchill [1958, p. 315 (11)] to eqns. (2.1) to (2.3) and using the property [Churchill 1958, p. 315 (13)], one obtains

$$\frac{1}{v} \frac{dv}{dt}(\lambda_n, t) = \lambda_n v(\lambda_n, t) - \alpha_n r(a) \,.\, A_1$$
$$+ \beta_n r(b) \,.\, A_2 \qquad \ldots(3.1)$$

with

$$v(\lambda_n, t) = g(\lambda_n) \quad \text{at } t = 0 \qquad \ldots(3.2)$$

and

$$v(\lambda_n, t) = h(\lambda_n) \quad \text{at } t = T_1. \qquad \ldots(3.3)$$

The equations (3.1) to (3.3) yields

$$h(\lambda_n) - g(\lambda_n) \exp(\lambda_n v T_1)$$
$$= \frac{\beta_n r(b) A_2 - \alpha_n r(a) A_1}{\lambda_n} [1 - \exp(-\lambda_n v T_1)]. \qquad \ldots(3.4)$$

Next applying the inverse transformation defined by Churchill [1958, p. 315 (12)] to eqn. (3.4), one obtains

$$H(x) = \sum_{n=1}^{\infty} g(\lambda_n) \exp(\lambda_n v T_1) \phi_n(x)$$
$$+ \sum_{n=1}^{\infty} \left\{ \frac{\{\beta_n r(b) A_2 - \alpha_n r(a) A_1\}}{\lambda_n} \right.$$
$$\left. \times [1 - \exp(-\lambda_n v T_1)] \right\} \phi_n(x) \qquad \ldots(3.5)$$

Divide now the range (a, b) into s equal intervals and let

$$G\left[k\frac{(b-a)}{s}\right] = G_k \qquad \text{...(3.6)}$$

and

$$H\left[k\frac{(b-a)}{s}\right] = H_k\,. \qquad \text{...(3.7)}$$

Using trapezoidal rule, one gets

$$g(\lambda_n) = \sum_{j=0}^{s} K_j\, G_j\, \phi_n\left[j\frac{(b-a)}{s}\right] . \, p\left[j\frac{(b-a)}{s}\right] \qquad \text{...(3.8)}$$

where

$$K_j = \begin{bmatrix} \dfrac{b-a}{s}, & j \neq s, \\ \dfrac{b-a}{2s}, & j = s. \end{bmatrix} \qquad \text{...(3.9)}$$

Then eqn. (3.5) becomes, for $0 \leq i \leq s$,

$$H_i = c_{i0} + \sum_{j=1}^{s} C_{ij}\, G_j \qquad \text{...(3.10)}$$

where

$$c_{i0} = \sum_{n=1}^{\infty} \{\beta_n\, r(b)\, A_2 - \alpha_n\, r(a)\, A_1\}$$

$$\times \left(1 - \exp(\lambda_n \nu T_1)\right) \phi_n\left[i\frac{(b-a)}{s}\right] \qquad \text{...(3.11)}$$

and

$$C_{ij} = \sum_{n=1}^{\infty} \exp(\lambda_n \nu T_1)\, K_j\, G_j\, \phi_n\left[j\frac{(b-a)}{s}\right]$$

$$\times\, p\left[j\frac{(b-a)}{s}\right] . \, \phi_n\left[i\frac{(b-a)}{s}\right] . \qquad \text{...(3.12)}$$

And hence the matrix notation of eqn. (3.10) is

$$H = c_0 + CG. \qquad \text{...(3.13)}$$

Similarly, the set of equations (2.6) to (2.10) gives

$$G = d_0 + DH, \qquad \text{...(3.14)}$$

where

$$d_0 = \begin{bmatrix} d_{00} \\ d_{10} \\ \vdots \\ d_{s0} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_{01} \ldots D_{0s} \\ \vdots \quad\quad \vdots \\ D_{s0} \ldots D_{ss} \end{bmatrix}, \qquad \ldots(3.15)$$

in which

$$d_{i0} = \sum_{n=1}^{\infty} \{\beta_n r(b) A_4 - \alpha_n r(a) A_3\} (1 - \exp(-\lambda_n \nu T_2))$$
$$\phi_n \left[i \frac{(b-a)}{s} \right], \qquad \ldots(3.16)$$

and

$$D_{ij} = \sum_{n=1}^{\infty} \exp(\lambda_n \nu T_2) K_j H_j \phi_n \left[j \frac{(b-a)}{s} \right]$$
$$\times p \left[j \frac{(b-a)}{s} \right] \phi_n \left[i \frac{(b-a)}{s} \right]. \qquad \ldots(3.17)$$

Solving equations (3.13) and (3.14), one gets

$$G = [I - DC]^{-1} [Dc_0 + d_0] \qquad \ldots(3.18)$$

and

$$H = [I - CD]^{-1} \ [Cd_0 + c_0], \qquad \ldots(3.19)$$

where I is the unit matrix.

From the above, it is obvious that the matrices C, D, c_0 and d_0 are known, and hence the functions $H(x)$ and $G(x)$ are known.

After calculating the values of the functions $G(x)$ and $H(x)$, the value of the function $V(x, t)$ can be calculated easily.

The function $V(x, t)$ for the time interval $(0, T_1)$ can be obtained from eqns. (2.1), (2.2), (2.4) and (2.5) as

$$V(x, t) = \sum_{n=1}^{\infty} g(\lambda_n) \exp(\lambda_n \nu t) \phi_n(x)$$
$$+ \sum_{n=1}^{\infty} \{\beta_n r(b) A_2 - \alpha_n r(a) A_1\}$$
$$\times \{1 - \exp[-\lambda_n \nu t]\} \phi_n(x), \qquad \ldots(3.20)$$

and the function $V(x, t)$ for the time interval $(T_1, T_1 + T_2)$ can be obtained from eqns. (2.6), (2.7), (2.9) and (2.10) as

$$V(x, t) = \sum_{n=1}^{\infty} h(\lambda_n) \exp(\lambda_n \nu J) \phi_n(x)$$
$$+ \sum_{n=1}^{\infty} \{\beta_n r(b) A_4 - \alpha_n r(a) A_3\}$$
$$\times \{1 - \exp[-\lambda_n \nu J]\} \phi_n(x). \qquad \ldots(3.21)$$

Equations (2.18) to (3.21) constitute the solution of the boundaiy value problem posed by eqns. (2.1) to (2.10).

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References

Churchill, R. V. (1958). Operational Mathematics, Second Edn. McGraw-Hill Book Co., Inc., New York.

Howard, J. R., and Sutton, A. E. (1970a). An analogue study of heat transfer through periodically contacting surfaces. *Int. J. Heat Mass Transfer*, **13**, 173–83.

——— (1970b). An analogue study of heat transfer through periodically contacting surfaces: The effect of thermal conductance. *Tech. Note MECH/28, The University of Aston in Birmingham, Birmingham.*

Reed, J. R., and Mullineux, G. (1973). Quasi-steady state solution of periodically varying phenomena. *Int. J. Heat Mass Transfer*, **16**, 2007–12.

Wankhede, P. C. (1975). Quasi-steady motion of a viscous fluid. *Presented at 62nd Session of the Indian Science Congress.*