ON SOME DOUBLE INTEGRALS INVOLVING MEIJER'S G-FUNCTION

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In the present paper the author has established two integral relations of Meijer's G-function of two variables. The results have been justified by illustrating some examples. Finally special cases have also been deduced for G-function of one variable.

1. INTRODUCTION

Recently Prasad and Ram (1973) have established two integrals which generalise the results of Dahiya (1971). The purpose of this paper is to establish two integrals relations of Meijer's G-function of two variables which are similar to the ones studied.

The symbols used have their usual meanings (*cf.* Agarwal 1965). We shall use the following formulae due to (Sneddon 1956):

$$\int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^v\, d\theta = \frac{\Gamma(v+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{v+1}\,\Gamma(\frac{1}{2}v+u+1)\,\Gamma(\frac{1}{2}v-u+1)},$$

$$\operatorname{Re}(v) > 0 \quad \ldots(1.1)$$

$$\int_0^{\pi/2} \cos 2u\theta\,(\cos\theta)^v\, d\theta = \frac{(-)^u\,\Gamma(v+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{v+1}\,\Gamma(\frac{1}{2}v+u+1)\,\Gamma(\frac{1}{2}v-u+1)}$$

$$= \frac{\pi\Gamma(v+1)}{2^{v+1}\,\Gamma(\frac{1}{2}v+u+1)\,\Gamma(\frac{1}{2}v-u+1)} \quad \ldots(1.2)$$

where u is an integer and $\operatorname{Re}(v) > 0$.

The result (1.2) is obtained by taking $(\pi/2-\theta)$ in place of θ from (1.1).

2. MAIN RESULTS

We give two integral relations of Meijer's G-function of two variables in the present paper.

$$\int_0^{\infty}\int_0^{\infty} \frac{y^{2v}\cos 2u\left(\tan^{-1}\dfrac{y}{x}\right)}{(x^2+y^2)^{v}}$$

$$\times\ G^{n,\,v_1,\,v_2,\,m_1,\,m_2}_{p,\,[t;\,t'],\,s,\,[q;\,q']}\left[\begin{array}{c|l} & (t_p) \\ \alpha\,(x^2+y^2)^{\rho-1}\,y^2 & (\gamma_t)\ ;\ (\gamma^1{}_{t'}) \\ \beta\,(x^2+y^2)^{\rho-1}\,y^2 & (\delta_s) \\ & (\beta_q)\ ;\ (\beta^1{}_{q'}) \end{array}\right]$$

$$\times f\,(x^2+y^2)\,dxdy$$

$$= \frac{\Gamma\left(\tfrac{1}{2}+u\right)\Gamma\left(\tfrac{1}{2}-u\right)}{4\sqrt{\pi}}\int_0^\infty G^{n+2,\,v_1,\,v_2,\,m_1,\,m_2}_{p+2,\,[t;\,t^1],\,s+2,\,[q,\,q^1]}$$

$$\times\left[\begin{array}{c|l} & \tfrac{1}{2}-v,\ -v,\ (\varepsilon_p) \\ \alpha z^{\rho} & (\gamma_t)\ ;\ (\gamma'_{t'}) \\ \beta z^{\rho} & 1+u+v,\ 1-u+v,\ (\delta_s) \\ & (\beta_q)\ ;\ (\beta'_{q'}) \end{array}\right] f\,(z)\,dz. \qquad \ldots(2.1)$$

$$\int_0^\infty\int_0^\infty \frac{x^{2v}\cos 2u\left(\tan^{-1}\dfrac{y}{x}\right)}{(x^2+y^2)^{v}}$$

$$\times\ G^{u,\,v_1,\,v_2,\,m_1,\,m_2}_{p,\,[t;\,t^1],\,s,\,[q,\,q']}\left[\begin{array}{c|l} & (t_p) \\ \alpha\,(x^2+y^2)^{\rho-1}\,x^2 & (\gamma_t)\ ;\ (\gamma'_{t'}) \\ \beta\,(x^2+y^2)^{\rho-1}\,x^2 & (\delta_s) \\ & (\beta_q)\ ;\ (\beta'_{q'}) \end{array}\right]$$

$$\times f\,(x^2+y^2)\ dxdy$$

$$= \frac{\sqrt{\pi}}{4}\int_0^\infty G^{\,u+2,\,v_1,\,v_2,\,m_1,\,m_2}_{\,p+2,\,[t\,;\,t'],\,s,\,[q\,;\,q']}$$

$$\times\left[\begin{array}{c|l} & \tfrac{1}{2}-v,\ -v,\ \ (\epsilon_p) \\ \alpha z^{\rho} & (\gamma_t),\ \ ;\ \ (\gamma'_{t'}) \\ \beta z^{\rho} & (\delta_s),\ \ 1+u+v,\ \ 1-u+v \\ & (\beta_q)\ \ ;\ \ (\beta'_{q'}) \end{array}\right] f\,(z)\,dz \qquad \ldots(2.2)$$

where

$$\operatorname{Re}(v) > 0$$

$$0 \le m_1 \le q,\quad 0 \le m_2 \le q',\quad 0 \le v_1 \le t,\quad 0 \le v_2 \le t',\quad 0 \le n \le p$$

$$p+q+s+t < 2\,(m_1+v_1+n)$$

$$p+q'+s+t' < 2\,(m_2+v_2+n)$$

$$|\arg x| < \pi\,[m_1+v_1+n-\tfrac{1}{2}(p+q+s+t)]$$

$$|\arg y| < \pi\,[m_2+v_2+n-\tfrac{1}{2}(p+q'+s+t')].$$

3. Proof

To obtain our first result we start with the following integral,

$$\int_0^{\infty} \cos 2u\theta \, (\sin\theta)^{2v} \; G^{n, \nu_1, \nu_2, m_1, m_2}_{p, [t;\, t'], s, [q, q']} \times \left[\begin{matrix} \alpha z^{\rho} \sin^2\theta \\ \beta z^{\rho} \sin^2\theta \end{matrix} \,\middle|\, \begin{matrix} (t_p) \\ (\gamma_t);\; (\gamma'_{t'}) \\ (\delta_s) \\ (\beta_q);\; (\beta'_{q'}) \end{matrix} \right] d\theta \qquad \ldots(3.1)$$

Now, replacing the *G*-function by its equivalent double integral in (3.1) from Agarwal (1965) and changing the order of integration we have,

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \phi\psi \, (\alpha z^{\rho})^{\xi} \, (\beta z^{\rho})^{\eta}$$

$$\times \int_0^{\pi/2} \cos 2u\theta \, (\sin\theta)^{2(v+\xi+\eta)} \, d\theta \, . \, d\xi d\eta \qquad \ldots(3.2)$$

Using the formula (1.1) in (3.2) the R.H.S. of (3.2) is

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \phi\psi \, (\alpha z^{\rho})^{\xi} \, (\beta z^{\rho})^{\eta}$$

$$\times \frac{\Gamma\{2(v+\xi+\eta)+1\}}{P\{v+u+1+\xi+\eta\}\Gamma\{1-u+v+\xi+\eta\} \, 2^{2(v+\xi+\eta)+1}} \, d\xi d\eta$$

where

$$\phi(\xi+\eta) = \frac{\prod_{j=1}^{m_1} \Gamma(1-\epsilon_j+\xi+\eta)}{\prod_{j=n+1}^{p} \Gamma(\epsilon_j-\xi-\eta) \prod_{j=1}^{s} \Gamma(\delta_j+\xi+\eta)}$$

$$\psi(\xi,\eta) = \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j-\xi) \prod_{j=1}^{\nu_1} \Gamma(\gamma_j+\xi) \prod_{j=1}^{m_2} P(\beta'_j-\eta) \prod_{j=1}^{\nu_2} \Gamma(\gamma'_j+\eta)}{\prod_{j=m_1+1}^{q} \Gamma(1-\beta_j+\xi) \prod_{j=\nu_1+1}^{t} \Gamma(1-\gamma_j-\xi) \prod_{j=1+m_2}^{q'} \Gamma(1-\beta'_j+\eta) \prod_{j=1+\nu_2}^{t'} \Gamma(1-\gamma'_j-\eta)}$$

Now on using the duplication formula of Gamma function,

$$2^{2z-1} = \frac{\sqrt{\pi}\,\Gamma(2z)}{\Gamma(z)\,\Gamma(z+\frac{1}{2})} \qquad \ldots(3.3)$$

the above integral becomes,

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}} \cdot \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \phi(\xi+\eta)\,\psi(\xi,\eta)\,(\alpha z^{\rho})^{\xi}\,(\beta z^{\rho})^{\eta}$$

$$\times \frac{\Gamma(v+\frac{1}{2}+\xi+\eta)\,\Gamma(v+1+\xi+\eta)}{\Gamma(v+u+1+\xi+\eta)\,\Gamma(v-u+1+\xi+\eta)}\, d\xi d\eta \qquad \ldots(3.4)$$

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}} \cdot G^{n+2,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p+2,\,[t;\,t'],\,s+2,\,[q;\,q']}$$

$$\times \left[\begin{array}{l|l} & \frac{1}{2}-v,\ -v,\ (\epsilon_p) \\ \alpha z^{\rho} & (\gamma_t)\ ,\ (\gamma'_{t'}) \\ \beta z^{\rho} & 1+u+v,\ 1-u+v,\ (\delta_s) \\ & (\beta_q)\ ,\ (\beta'_{q'}) \end{array}\right] \qquad \ldots(3.5)$$

provided that, $\mathrm{Re}(v) > 0$, $|\arg\alpha| < \frac{1}{2}\lambda\pi$, $\lambda > 0$.

On putting $z = r^2$ in (3.5) and then multiplying both sides by $f(r)\,dr$ and integrating between $(0, \infty)$, we have,

$$\int_0^{\infty} r f(r^2)\, dr \int_0^{\pi/2} \cos 2u\theta\,(\sin\theta)^{2v} \,.\, G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,[t;\,t'],\,s,\,[q;\,q']}$$

$$\times \left[\begin{array}{l|l} & (t_p) \\ \alpha r^{2\rho}\sin^2\theta & (\gamma_t)\ ;\ (\gamma'_{t'}) \\ \beta r^{2\rho}\sin^2\theta & (\delta_s) \\ & (\beta_q)\ ;\ (\beta'_{q'}) \end{array}\right] dr$$

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}} \int_0^{\infty} r f(r^2)\,.\, G^{n+2,\,v_1,\,v_2,\,m_1,\,m_2}_{p+2,\,[t;\,t'],\,s+2,\,[q;q']}$$

$$\times \left[\begin{array}{l|l} & \frac{1}{2}-v,\,-v,\,(\epsilon_p) \\ \alpha r^{2\rho} & (\gamma_t)\ ;\ (\gamma'_{t'}) \\ \beta r^{2\rho} & 1+u+v,\ 1-u+v,\ (\delta_s) \\ & (\beta_q)\ ;\ (\beta'_{q'}) \end{array}\right] ds \qquad \ldots(3.6)$$

On putting $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\theta = \tan^{-1} y/x$ in the L.H.S. of (3.6) and simplifying further we obtain the result,

$$\int_0^\infty \cos 2u \left(\tan^{-1} \frac{y}{x}\right) \frac{y^{2v}}{(x^2+y^2)^v}$$

$$\times\; G^{n,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p,\, [t:\, t'],\, \;,\, [q;\, q']} \left[\begin{array}{c|l} & (\varepsilon_p) \\ \alpha (x^2+y^2)^{\rho-1} y^2 & (\gamma_t);\ (\gamma'_{t'}) \\ \beta (x^2+y^2)^{\rho-1} y^2 & (\delta_s) \\ & (\beta_q);\ \ (\beta'_{q}) \end{array} \right] f(x^2+y^2)\, dxdy$$

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{4\sqrt{\pi}} \int_0^\infty G^{n+2,\, \nu_1,\, \nu_2,\, m_1,\, m_2}_{p+2,\, [t;\, t'],\, s+2,\, [q;\, q']}$$

$$\times \left[\begin{array}{c|l} & \frac{1}{2}-u,\ -v,\ (\epsilon_p) \\ \alpha z^\rho & (\gamma_t);\ (\gamma'_{t'}) \\ \beta z^\rho & 1+u+v,\ 1-u+v,\ (\delta_s) \\ & (\beta_q);\ \ (\beta'_{q'}) \end{array} \right] f(z)\, dz$$

provided that,

$$0 \le m_1 \le q, \quad 0 \le m_2 \le q', \quad 0 \le V_1 \le t, \quad 0 \le \nu_2 \le t', \quad 0 \le n \le p$$

$$p + q + s + t < 2(m_1 + \nu_1 + n)$$

$$p + q' + s + t' < 2(m_2 + \nu_2 + n)$$

$$|\arg x| < \pi\,[m_1 + \nu_1 + n - \tfrac{1}{2}(p + q + s + t)]$$

$$|\arg y| < \pi\,[m_2 + \nu_2 + n - \tfrac{1}{2}(p + q' + s + t')]$$

and

$$\operatorname{Re}(v) > 0.$$

Similarly, the result (2.2) can also be obtained by using (1.2).

4. Example

As an illustration, we have below some special cases of (2.1) and (2.2).

Let us take $f(z) = z^{\sigma-1} W_{k,\mu}(z)\, W_{-k,\mu}(z)$ in (2.1) and making use of the known result (Erdelyi 1953) we obtain,

$$\int_0^\infty \int_0^\infty y^{2v} \cos 2u\left(\tan^{-1}\frac{y}{x}\right)(x^2+y^2)$$

$$\times\ G^{n,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p,\,[t;\,t'],\,s,\,[q;\,q']}\left[\begin{array}{c|l} & (\epsilon_p) \\ \alpha(x^2+y^2)\,y^2 & (\gamma_t);\quad (\gamma'_{t'}) \\ \beta(x^2+y^2)\,y^2 & (\delta_s) \\ & (\beta_q);\ (\beta'_{q'}) \end{array}\right]$$

$$\times\ W_{k,\,\mu}(x^2+y^2)\,W_{-k,\,\mu}(x^2+y^2)\,dxdy$$

$$= \frac{2^{\sigma-1}\,\Gamma(\tfrac{1}{2}+u)\,\Gamma(\tfrac{1}{2}-u)}{4\pi}\ G^{n+6,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p+6,\,[t;\,t'],\,s+4,\,[q;\,q']}$$

$$\times\left[\begin{array}{c|l} & \tfrac{1}{2}-u,\ -v,\ \tfrac{1}{2}-\dfrac{\sigma}{2}-\mu,\ \dfrac{1-\sigma}{2}+\mu,\ \dfrac{1-\sigma}{2},\ -\dfrac{\sigma}{2},\ (\epsilon_p) \\ 4\alpha & (\gamma_t)\ ;\quad (\gamma'_{t'}) \\ 4\beta & (\delta_s),\ \ 1+u+v,\ 1-u+v,\ 1+\dfrac{\sigma}{2}+k,\ 1+\dfrac{\sigma}{2}-k \\ & (\beta_q)\ ;\quad (\beta'_{q'}) \end{array}\right] \qquad \text{...(4.1)}$$

Similarly, if we substitute the above value of $f(z)$ in (2.2), we get,

$$\int_0^\infty \int_0^\infty x^{2v} \cos 2u\left(\tan^{-1}\frac{y}{x}\right)(x^2+y^2)^{1-v}$$

$$\times\ G^{n,\,\nu_1,\,\nu_2,\,m_2,\,m_2}_{p,\,[t;\,t'],\,s,\,[q;\,q']}\left[\begin{array}{c|l} & (\epsilon_p) \\ \alpha(x^2+y^2)\,x^2 & (\gamma_t)\ ;\ (\gamma'_{t'}) \\ \beta(x^2+y^2)\,x^2 & (\delta_s) \\ & (\beta_q)\ ;\ (\beta'_{q'}) \end{array}\right]$$

$$\times\ W_{k,\,\mu}(x^2+y^2)\,W_{-k,\,\mu}(x^2+y^2)\,dxdy$$

$$= \tfrac{1}{2}\ G^{n+6,\,\nu_1,\,\nu_2,\,m_1,\,m_2}_{p+6,\,[t;\,t'],\,s+4,\,[q;\,q']}\left[\begin{array}{l} \tfrac{1}{2}-v,\ -v,\ -\tfrac{1}{2}-\mu,\ -\tfrac{1}{2}+\mu,\ -\tfrac{1}{2},\ -1,\ (\varepsilon_p) \\ (\gamma_t)\ ;\quad (\gamma'_{t'}) \\ (\delta_s),\ \ 1+u+v,\ 1-u+v,\ 2+k,\ 2-k \\ (\beta_q)\ ;\quad (\beta'_{q'}) \end{array}\right] \qquad \text{...(4.2)}$$

provided that

$$|\arg \alpha| < \tfrac{1}{2}\lambda\pi, \quad \lambda > 0, \quad A > 0$$

$$0 \le m_1 \le q, \quad 0 \le m_2 \le q', \quad 0 \le v_1 \le t, \quad 0 \le v_2 \le t', \quad 0 \le n \le p$$

$$p + q + s + t < 2(m_1 + v_1 + n)$$

$$p + q' + s + t' < 2(m_2 + v_2 + n)$$

$$|\arg x| < \pi[m_1 + v_1 + n - \tfrac{1}{2}(p + q + s + t)]$$

$$|\arg y| < \pi[m_2 + v_2 + n - \tfrac{1}{2}(p + q' + s + t')]$$

and

$$\operatorname{Re}(v) > 0.$$

Corresponding results for H-function of two variables can also be obtained which would generalise the results (2.1) and (2.2).

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