AN EXACT SOLUTION OF THE TRANSIENT FORCED CONVECTION ENERGY EQUATION FOR TIMEWISE LINEAR VARIATION OF INLET TEMPERATURE IN A CIRCULAR PIPE

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An exact solution of the transient forced convection energy equation of a viscous incompressible fluid with fully developed flow in a circular pipe is obtained in the present investigation when the inlet temperature varies linearly with time. The Laplace transform technique has been used to obtain the solution and an interpretation of the case of laminar flows is given.

NOMENCLATURE

$\bar{T}$	= temperature
c_p	= specific heat at constant pressure
a	= pipe radius
κ	= thermal conductivity
$\bar{t}$	= time
u	= velocity component in $\bar{z}$-direction
$\bar{u}$	= average velocity or mean velocity
$\bar{r}, \phi, \bar{z}$	= cylindrical polar coordinates ($\bar{z}$-flow direction)
ρ	= liquid density
ν	= kinematic coefficient of viscosity
R	= Reynolds number $\left(= \frac{a\bar{u}}{\nu}\right)$
P	= Prandtl number $\left(= \frac{\rho \nu c_p}{\kappa}\right)$
T_0, T_1, T_2	= known constant temperatures.

1. INTRODUCTION

The study of unsteady forced convection heat transfer in tubes and ducts has recently become of greater importance in connection with the control of modern high performance heat transfer devices. Literature on thermal transient problems is limited but increasing. Some of the important contributions are listed in the references (Kakac 1965, Ozisik 1968). In solutions of the problems of transient forced convection in laminar flow, it has usually been assumed that the inlet temperature of the fluid is constant across the flow with a specified time-wise variation of wall temperature, wall heat flux or internal heat generation. There are also some works on the thermal transient problems in heat exchangers. Recently, Kakac and Yener (1973) have obtained an exact solution of the transient forced convection energy equation for laminar as well as turbulent flow in a parallel plate channel under a prescribed boundary condition with an inlet temperature which varies sinusoidally with time. They have also given experimental results for the lowest eigenvalue for turbulent flow.

In the present paper a solution of the transient forced convection energy equation is obtained for laminar flow in a circular pipe under a prescribed boundary condition when the inlet temperature varies linearly with time and an interpretation of the case of laminar flows is given.

2. Formulation of the Problem

We consider the steady laminar flow of a viscous incompressible fluid through a circular pipe of radius a. The fluid entering the pipe has a temperature which is spatially uniform across the entrance section but varies linearly with time. Therefore we can write the inlet condition as

$$\bar{T}(\bar{r}, 0, \bar{t}) = T_0 + T_1\left(\frac{\nu\bar{t}}{a^2}\right). \qquad \text{...(2.1)}$$

The unsteady energy equation for a fully developed flow in a circular pipe is given by

$$\frac{\partial \bar{T}}{\partial \bar{t}} + u\frac{\partial \bar{T}}{\partial \bar{z}} = \frac{\kappa}{\rho c_p}\left(\frac{\partial^2 \bar{T}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{T}}{\partial \bar{r}}\right). \qquad \text{...(2.2)}$$

The inlet and the boundary conditions of the problem are as follows:

$$\bar{T} = T_0 + T_1\left(\frac{\nu\bar{t}}{a^2}\right) \quad \text{when} \quad \bar{z} = 0, \qquad \text{...(2.3)}$$

$$\bar{T} \text{ is finite at } \bar{r} = 0, \quad \bar{T} = T_2 \text{ at } \bar{r} = a \quad (\bar{t} > 0). \qquad \text{...(2.4)}$$

The equation (2.2) is subjected to the following restrictions:

(*a*) Fully developed laminar velocity profile in the circular pipe.

(*b*) Frictional dissipation of energy is negligible.

(*c*) Axial conduction is negligible as compared to bulk transport in the $\bar{z}$-direction. This is a reasonable assumption when Péclét number exceeds 100 (Kakac and Yener 1973).

(*d*) Liquid property variations are also neglected.

(*e*) Thermal resistance of the pipe wall is negligible.

Further, to simplify the method of analysis the case of constant velocity will be treated here and for this purpose we substitute $\bar{u}$ for the velocity profile in (2.2).

We now introduce the following non-dimensional quantities:

$$\theta = \frac{\bar{T} - T_0}{T_1}, \qquad z = \frac{\bar{z}}{a}, \qquad r = \frac{\bar{r}}{a},$$

$$t = \frac{\nu \bar{t}}{a^2}, \qquad \theta_0 = \frac{T_2 - T_0}{T_1}.$$

Equation (2.2) then becomes

$$\frac{\partial \theta}{\partial t} + R \frac{\partial \theta}{\partial z} = \frac{1}{P} \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right). \qquad \ldots(2.5)$$

The inlet and the boundary conditions reduce to

$$\theta = t \quad \text{when} \quad z = 0, \qquad \ldots(2.6)$$

$$\theta \text{ is finite at } r = 0, \; \theta = \theta_0 \text{ at } r = 1 \;\; (t > 0). \qquad \ldots(2.7)$$

3. Solution

In obtaining $\theta(r, z, t)$ we assume that

$$\theta(r, z, t) = \theta_1(r, z) + t\theta_2(r, z), \qquad \ldots(3.1)$$

where the new functions θ_1 and θ_2 satisfy the following problems:

$$\left.\begin{aligned} &PR \frac{\partial \theta_2}{\partial z} = \left(\frac{\partial^2 \theta_2}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_2}{\partial r} \right), \\ &\theta_2 = 1 \text{ when } z = 0, \\ &\theta_2 \text{ is finite at } r = 0, \;\; \theta_2 = 0 \;\; \text{at} \;\; r = 1. \end{aligned}\right\} \qquad \ldots(3.2)$$

$$PR \frac{\partial \theta_1}{\partial z} + P\theta_2 = \left(\frac{\partial^2 \theta_1}{\partial r^2} + \frac{1}{r} \frac{\partial \theta_1}{\partial r} \right) \qquad \ldots(3.3)$$

$$\theta_1 = 0 \quad \text{when} \quad z = 0 \qquad \ldots(3.4)$$

$$\theta_1 \text{ is finite at } r = 0, \;\; \theta_1 = \theta_0 \;\; \text{at} \;\; r = 1. \qquad \ldots(3.5)$$

The solution of (3.2) is given by

$$\theta_2(r, z) = 2 \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp\left(-\frac{\alpha_n^2}{PR} z \right) \qquad \ldots(3.6)$$

where the α_n $(n = 1, 2, 3, \ldots)$ are the positive roots of $J_0(\alpha) = 0$.

Then the solution of (3.3) under the conditions (3.4) and (3.5) is given by

$$\theta_1(r, z) = \theta_0\left[1 - 2\sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp\left(-\frac{\alpha_n^2}{PR} z\right)\right]$$

$$-\frac{2z}{R}\sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp\left(-\frac{\alpha_n^2}{PR} z\right). \qquad \text{...(3.7)}$$

Thus

$$\theta(r, z, t) = \theta_0 + 2\left(t - \frac{z}{R} - \theta_0\right)\sum_{n=1}^{\infty} \frac{J_0(r\,\alpha_n)}{\alpha_n J_1(\alpha_n)} X_n, \qquad \text{...(3.8)}$$

where

$$X_n = \exp\left(-\frac{\alpha_n^2}{PR} z\right).$$

4. Discussion

When the boundary condition on the wall for $\theta(r, z, t)$ is homogeneous, that is, when θ_0 is zero, then

$$\theta(r, z, t) = 2\left(t - \frac{z}{R}\right)\sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n J_1(\alpha_n)} \exp\left(-\frac{\alpha_n^2}{PR} z\right). \qquad \text{...(4.1)}$$

This shows that each mode of the temperature field decays exponentially along the pipe.

In many applications heat transfer in regions away from the inlet is of interest; for such situations only the first term in the series need to be considered and from eqn. (4.1), we then get

$$\theta(r, z, t) = 2\left(t - \frac{z}{R}\right) \frac{J_0(r\alpha_1)}{\alpha_1 J_1(\alpha_1)} \exp\left(-\frac{\alpha_1^2}{PR} z\right). \qquad \text{...(4.2)}$$

Then the temperature at any r, say $r = 0$ (axis of the pipe), is given by

$$\theta(0, z, t) = 2\left(t - \frac{z}{R}\right) \frac{1}{\alpha_1 J_1(\alpha_1)} \exp\left(-\frac{\alpha_1^2}{PR} z\right). \qquad \text{...(4.3)}$$

$\theta(0, z, t)$ at various points along the pipe have been presented in graphical forms (Figs. 1 and 2) for $R = 13000$, 20000, 27000 when $P = 0{\cdot}73$, $t = 2$

(Fig. 1) and for $P = 4, 9, 15$ when $R = 2140$, $t = 2$ (Fig. 2). From these two figures we observe that $\theta(o, z, t)$ increases as the Prandtl number and the Reynolds number increase.

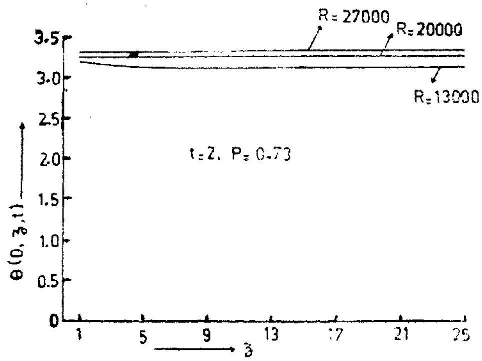

FIG. 1

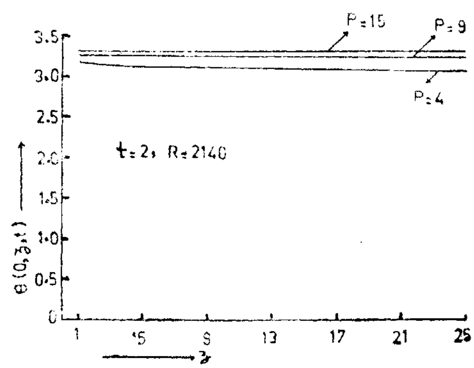

FIG. 2

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