ON ALMOST SASAKIAN MANIFOLD

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In this paper we have studied Lie derivatives and almost analytic vector fields in an almost Sasakian manifold.

1. INTRODUCTION

Let us consider an $n (= 2m + 1)$ dimensional real differentiable manifold V_n of class C^∞. Let there exist in V_n a C^∞ vector valued linear function F, a C^∞ vector field T and a C^∞1-form A satisfying (Mishra 1970).

$$\bar{\bar{X}} + X = A(X)\,T,\ \bar{X} \overset{\text{def}}{=} F(X) \qquad \text{...(1.1)}$$

which implies

$$A(\bar{X}) = 0,\ \bar{T} = 0,\ A(T) = 1 \text{ and rank } (F) = n - 1 \qquad \text{...(1.2)}$$

for arbitrary vector field X in V_n. Then V_n is called an almost contact manifold and the structure (F, T, A) is called an almost contact structure.

Let there be a metric tensor G in an almost contact manifold satisfying

$$G(\bar{X}, \bar{Y}) = G(X, Y) - A(X)\,A(Y). \qquad \text{...(1.3)}$$

Then V_n is called an almost contact metric manifold or an almost Grayan manifold and the structure (F, T, A, G) is called an almost Grayan structure.

Put

$$'F(X, Y) = G(\bar{X}, Y). \qquad \text{...(1.4)}$$

Then $'F$ is skew-symmetric

$$'F(X, Y) = -\,'F(Y, X). \qquad \text{...(1.5)}$$

In an almost Grayan manifold, let

$$2'F(X, Y) = (D_X A)(Y) - (D_Y\ A)\ (X) = (d\,A)(X, Y), \qquad \text{...(1.6)}$$

where D is a Riemannian connexion. Then V_n is called an almost Sasakian manifold and the structure (F, T, A, G, D) is called on almost Sasakian structure (Sasaki and Hatakayama 1962). It may be noted that in an almost Sasakian manifold $'F$ is closed:

$$(D_X{}'F)\ (Y, Z) + (D_Y{}'F)(Z, X) + (D_Z{}'F)(X, Y) = 0. \qquad \ldots(1.7)$$

Definition 1.1—A vector T is said to be a killing vector if it satisfies

$$(D_X A)(Y) + (D_Y A)(X) = 0, \qquad \ldots(1.8)$$

and T is said to be a harmonic vector if it satisfies

$$(D_X A)(Y) - (D_Y A)(X) = 0, \qquad \ldots(1.9a)$$

and

$$D_X T = 0. \qquad \ldots(1.9b)$$

2. Lie Derivatives

Lie derivatives along any vector V in V_n are of a type preserving function $\underset{V}{L}$ such that (Yano 1965)

$$\underset{V}{L} f = Vf, \quad f \text{ belongs to the field} \qquad \ldots(2.1a)$$

$$\underset{V}{L} X = [V, X] \qquad \ldots(2.1b)$$

$$(\underset{V}{L} A)(X) = V(A(X)) - A([V, X]) \qquad \ldots(2.1c)$$

$$(\underset{V}{L} P)\{A_1, A_2, \ldots, A_r, X_1, X_2, \ldots X_s\} = V\{P\{A_1, \ldots, X_s\}$$
$$- P\{\underset{V}{L} A_1, \ldots, X_s\} \ldots - P\{A_1, A_2, \ldots, A_r, [V, X_1], \ldots X_s\}$$
$$\ldots - P\{A_1, A_2, \ldots, [V, X_s]\}, \qquad \ldots(2.1d)$$

where $[\ ,]$ denotes Lie bracket and P is a tensor of type (r, s).

Theorem 2.1—In an almost Sasakian manifold V_n with T as a killing vector, we have

$$(\underset{X}{L} A)\ (Y) + (\underset{Y}{L} A)\ (X) = A\ (D_X Y + D_Y X). \qquad \ldots(2.2)$$

Proof: From equation (2.1c), we have

$$(\underset{X}{L} A)\ (Y) = (D_X A)\ (Y) + A(D_Y X). \qquad \ldots(2.3a)$$

Interchanging X and Y in the above equation, we have

$$(\underset{Y}{L} A)\ (X) = (D_Y A)\ (X) + A\,(D_X Y). \qquad \ldots(2.3b)$$

Adding (2.3 a) and (2.3 b) and using (1.8), we get (2.2).

Corollary 2.1—In an almost Sasakian manifold V_n with T as a killing vector, we have

$$(\underset{\bar{X}}{L} A)\ (\bar{Y}) + (\underset{\bar{Y}}{L} A)\ (\bar{X}) = A\ \big((D_{\bar{X}} F)\ (Y) + (D_{\bar{Y}} F)\ (X)\big). \qquad \ldots(2.4)$$

PROOF: Barring X and Y in (2.2) and using (1.1) and (1.2), we get (2.4).

Theorem 2.2—In an almost Sasakian manifold with T as a harmonic vector, we have

$$(\underset{X}{L} A)\ (Y) - (\underset{Y}{L} A)\ (X) + A\,([X, Y]) = 0. \qquad \ldots(2.5)$$

PROOF: Subtracting (2.3b) from (2.3a) and using (1.9a), we get (2.5).

3. NIJENHUIS TENSOR

A vector valued bilinear function N given by (Mishra 1972)

$$N\,(X, Y) \overset{\text{def}}{=} D_{\bar{X}}\,\bar{Y} - D_{\bar{Y}}\,\bar{X} - \overline{D_{\bar{X}}\,Y} + \overline{D_Y\,\bar{X}} - \overline{D_X\,\bar{Y}} + \overline{D_{\bar{Y}}\,X} - D_X\,Y + D_Y\,X + A\,(D_X\,Y - D_Y\,X)\,T \qquad \ldots(3.1)$$

is called the Nijenhuis tensor of type (1, 2) with respect to the connexion D.

Corresponding to the Nijenhuis tensor of an almost complex space there are other tensors in V_n given by Mishra (1972). A tensor P of type (0, 2) which is a scalar valued bilinear function, Q a tensor of type (1, 1) which is a vector-valued linear function and R which is a 1-form, are given by

$$P\,(X, Y) \overset{\text{def}}{=} (D_Y\,A)\,(\bar{X}) - (D_X\,A)\,(\bar{Y}) + (D_{\bar{Y}}\,A)\,(X) - (D_{\bar{X}}\,A)\,(Y), \qquad \ldots(3.2)$$

$$Q\,(X) \overset{\text{def}}{=} D_T\,\bar{X} - D_{\bar{X}}\,T + \overline{D_X\,T} - \overline{D_T\,X} \qquad \ldots(3.3)$$

and

$$R\,(X) \overset{\text{def}}{=} (D_X\ A)\,(T) - (D_T\,A)\,(X). \qquad \ldots(3.4)$$

We shall now obtain some properties of $N\,(X, Y)$, $P\,(X, Y)$, $Q\,(X)$ and $R\,(X)$ with respect to the Lie derivatives.

Theorem 3.1—We have in an almost Sasakian manifold V_n,

$$(\underset{\bar{X}}{L} F)(Y) + (\underset{X}{L} F)(\bar{Y}) = N(X, Y) + A(Y)[X, T] + \big((D_X A)(Y) + A(D_Y X)\big) T. \quad ...(3.5)$$

PROOF: We have

$$(\underset{X}{L} F)(Y) = D_X \bar{Y} - D_{\bar{Y}} X - \overline{D_X Y} + \overline{D_Y X}. \quad ...(3.6)$$

Barring X and Y alternately in the above equation, we get

$$(\underset{\bar{X}}{L} F)(Y) = D_{\bar{X}} \bar{Y} - D_{\bar{Y}} \bar{X} - \overline{D_{\bar{X}} Y} + \overline{D_Y \bar{X}} \quad ...(3.7a)$$

and

$$(\underset{X}{L} F)(\bar{Y}) = D_X \bar{\bar{Y}} - D_{\bar{\bar{Y}}} X - \overline{D_X \bar{Y}} + \overline{D_{\bar{Y}} X} \quad ...(3.7b)$$

respectively.

Adding (3.7*a*) and (3.7*b*) and using (1.1) and (3.1), we get (3.5).

Remark 3.1: In an almost Sasakian manifold V_n, it can be easily verified that

$$(\underset{\bar{X}}{L} F)(Y) - \overline{(\underset{X}{L} F)(Y)} = [\bar{X}, \bar{Y}] + \overline{\overline{[X, Y]}} - \overline{[\bar{X}, Y]} - \overline{[X, \bar{Y}]}. \quad ...(3.8)$$

On the consequence of (3.1), (3.8) can be written as

$$(\underset{\bar{X}}{L} F)(Y) - \overline{(\underset{X}{L} F)(\bar{Y})} = N(X, Y). \quad ...(3.9)$$

Corollary 3.1—In almost Sasakian manifold, we have

$$(\underset{\bar{X}}{L} F)(T) = N(X, T) + [X, T] - A([X, T]) T. \quad ...(3.10)$$

PROOF: Putting T for Y (3.5) and using (1.2), we get (3.10).

Corollary 3.2—We have in V_n

$$(\underset{T}{L} F)(\bar{Y}) = N(T, Y) + \big((D_T A)(Y) + A(D_Y T)\big) T. \quad ...(3.11)$$

PROOF: Putting T for X in (3.5) and using (1.2) and the property of Lie brackets, we get (3.11).

Corollary 3.3—In an almost Sasakian manifold with T as harmonic vector, we have

$$(\underset{T}{L} F)(\bar{Y}) = N(T, Y). \quad ...(3.12)$$

PROOF: Putting T for X in (1.9*a*), we get

$$(D_T A)(Y) - (D_Y A)(T) = 0.$$

Using this equation and (1.2) in (3.11), we get (3.12).

Theorem 3.2—In an almost Sasakian manifold V_n, we have

$$P(X, Y) = (\underset{Y}{L} A)(\bar{X}) + (\underset{\bar{Y}}{L} A)(X) - (\underset{X}{L} A)(\bar{Y}) - (\underset{\bar{X}}{L} A)(Y)$$

$$+ A\left(D_{\bar{Y}} X - D_{\bar{X}} Y + (D_Y F)(X) - (D_X F)(Y)\right). \quad ...(3.13)$$

PROOF: Barring Y in (2.3*a*), we get

$$(\underset{X}{L} A)(\bar{Y}) = (D_X A)(\bar{Y}) + A(D_{\bar{Y}} X). \quad ...(3.14a)$$

Barring X in (2.3*a*) and using (1.1) and (1.2), we have

$$(\underset{\bar{X}}{L} A)(Y) = (D_{\bar{X}} A)(Y) + A\left((D_Y F)(X)\right). \quad ...(3.14b)$$

Interchanging X and Y in (3.14*a*) and (3.14*b*), we get

$$(\underset{Y}{L} A)(\bar{X}) = (D_Y A)(\bar{X}) + A(D_{\bar{X}} Y) \quad ...(3.14c)$$

and

$$(\underset{\bar{Y}}{L} A)(X) = (D_{\bar{Y}} A)(X) + A\left((D_X F)(Y)\right) \quad ...(3.14d)$$

respectively.

Subtracting (3.14*a*) and (3.14*b*) from the addition of (3.14*c*) and (3.14*d*) and using (3.2), we get (3.13).

Corollary 3.4—We have in V_n.

$$P(X, T) = (\underset{T}{L} A)(\bar{X}) - (\underset{\bar{X}}{L} A)(T) + A\left((D_T F)(X) - D_{\bar{X}} T\right.$$

$$\left. - (D_X F)(T)\right). \quad ...(3.15)$$

PROOF: Putting T for Y in (3.13) and using (1.2), we get (3.15).

Theorem 3.3—We have in an almost Sasakian manifold V_n,

$$Q(X) = (\underset{T}{L} F)(X). \quad ...(3.16)$$

PROOF: Putting T for X and X for Y in (3.6), we get

$$(\underset{T}{L} F)(X) = D_T \bar{X} - D_{\bar{X}} T - \overline{D_T X} + \overline{D_X T}.$$

Comparing this equation with (3.3), we get (3.16).

Theorem 3.4—We have in almost Sasakian manifold V_n

$$R(X) + (\underset{T}{L}\, A)(X) = 0. \qquad \ldots(3.17)$$

PROOF: Putting T for Y in (2.3*b*), we get

$$(\underset{T}{L}\, A)(X) = (D_T\, A)(X) + A(D_X T). \qquad \ldots(3.18)$$

Using (1.2) in (3.18) and comparing the result with equation (3.4), we get (3.17).

Theorem 3.5—Let us put

$$M(X, Y) = D_{\bar{X}}\, \bar{Y} + \overline{\overline{D_X\; Y}} - \overline{D_{\bar{X}}\; Y} - \overline{D_X\, \bar{Y}}. \qquad \ldots(3.19)$$

Then we have

$$(\underset{\bar{X}}{L}\, F)(Y) - \overline{(\underset{X}{L}\, F)(Y)} = M(X, Y) - M(Y, X). \qquad \ldots(3.20)$$

PROOF: Barring (3.6), we get

$$\overline{(\underset{X}{L}\, F)(Y)} = \overline{D_X\, \bar{Y}} - \overline{D_{\bar{Y}}\; X} - \overline{\overline{D_X\; Y}} + \overline{\overline{D_Y\; X}}. \qquad \ldots(3.21)$$

Subtracting (3.21) from (3.7*a*) and using (3.19), we get (3.20).

Theorem 3.6—In an almost Sasakian manifold, we have

$$(\underset{X}{L}\, 'F)(Y, Z) + (\underset{Y}{L}\, 'F)(Z, X) + (\underset{Z}{L}\, 'F)(X, Y) = {'F}(X, [Y, Z])$$

$$+\ {'F}(Y, [Z, X]) + {'F}(Z, [X, Y]). \qquad \ldots(3.22)$$

PROOF: On the consequence of (2.1*d*), we get

$$(\underset{X}{L}\, 'F)(Y, Z) = (D_X\, 'F)(Y, Z) + {'F}(D_X\, Y, Z) + {'F}(Y, D_X\, Z)$$

$$-\ {'F}([X, Y], Z) - {'F}(Y, [X, Z]). \qquad \ldots(3.23a)$$

By cyclic permutations of X, Y, Z in (3.23*a*), we get two other equations:

$$(\underset{Y}{L}\, 'F)(Z, X) = (D_Y\, 'F)(Z, X) + {'F}(D_Y\, Z, X) + {'F}(Z, D_Y\, X)$$

$$-\ {'F}([Y, Z], X) - {'F}(Z, [Y, X]) \qquad \ldots(3.23b)$$

and

$$(\underset{Z}{L}\, 'F)(X, Y) = (D_Z\, 'F)(X, Y) + {'F}(D_Z\, X, Y) + {'F}(X, D_Z\, Y)$$

$$-\ {'F}([Z, X], Y) - {'F}(X, [Z, Y]). \qquad \ldots(3.23c)$$

Adding (3.23*a*), (3.23*b*) and (3.23*c*) and using (1.7), we get (3.22).

4. Almost Analytic Vector Fields

Definition 4.1—A vector V is said to be contravariant almost analytic if it satisfies

$$(\underset{V}{L} F)(X) = 0 \qquad \text{...(4.1}a\text{)}$$

and

$$(\underset{V}{L} A)(X) = 0. \qquad \text{...(4.1}b\text{)}$$

Definition 4.2—A vector field V is said to be strictly almost analytic if both V and $\bar{V}$ are contravariant almost analytic, that is, if (4.1) and

$$(\underset{\bar{V}}{L} F)(X) = 0 \qquad \text{...(4.2}a\text{)}$$

and

$$(\underset{\bar{V}}{L} A)(X) = 0, \qquad \text{...(4.2}b\text{)}$$

are satisfied.

Theorem 4.1—The condition that a vector V be contravariant almost analytic is

$$\overline{[V, X]} = [V, \bar{X}] \text{ and } (D_V A)(X) + A(D_X V) = 0. \qquad \text{...(4.3)}$$

Proof: We have

$$(\underset{V}{L} F)(X) = [V, \bar{X}] - \overline{[V, X]} \qquad \text{...(4.4}a\text{)}$$

and

$$(\underset{V}{L} A)(X) = (D_V A)(X) + A(D_X V). \qquad \text{...(4.4}b\text{)}$$

By virtue of (4.1a) and (4.1b), we see that (4.4a) and (4.4b) vanish. Hence we have (4.3).

Corollary 4.1—We have in an almost Sasakian manifold

$$\overline{[V, T]} = 0 \text{ and } (D_V A)(T) + A(D_T V) = 0. \qquad \text{...(4.5)}$$

Proof: Putting T for X in (4.3) and using (1.2), we get (4.5).

Remark 4.1—If V is contravariant almost analytic, then it will be also strictly almost analytic but converse is not necessarily true.

Theorem 4.2—The condition that a vector field V be contravariant strictly almost analytic is

$$[V, \bar{X}] + [\bar{V}, \bar{X}] + V(A(X)) + \bar{V}(A(X)) = A([V, X]) + A([\bar{V}, X])$$
$$+ \overline{[V, X]} + \overline{[\bar{V}, X]}. \qquad \text{...(4.6)}$$

PROOF: Barring V in (4.4a), we get

$$(\underset{\bar{V}}{L} F)(X) = [\bar{V}, \bar{X}] - \overline{[\bar{V}, X]}. \qquad \text{...(4.7}a\text{)}$$

Again barring V in (2.1c), we get

$$(\underset{\bar{V}}{L} A)(X) = \bar{V}\,(A(X)) - A([\bar{V}, X]). \qquad \text{...(4.7}b\text{)}$$

Now adding (2.1c), (4.4a), (4.7a) and (4.7b) and using (4.1) and (4.2), we get (4.6).

Theorem 4.3—If a vector field V be contravariant strictly almost analytic in an almost Sasakian manifold, we have

$$A([V, T] + [\bar{V}, T]) = \overline{[T, V]} + [T, \bar{V}]. \qquad \text{...(4.8)}$$

PROOF: Putting T for X in (4.6) and using (1.2), we get (4.8).

Theorem 4.4—The condition that T be contravariant strictly almost analytic in an almost Sasakian manifold is

$$[T, \bar{X}] + (D_T A)(X) = \overline{[T, X]} - A(D_X T). \qquad \text{...(4.9)}$$

PROOF: Putting T for V in (4.6) and using (1.2), we get (4.9).

Theorem 4.5—In an almost Sasakian manifold V_n, we have

$$(\underset{\bar{V}}{L} A)(X) + (\underset{\bar{V}}{L} F)(X) - \overline{(\underset{V}{L} A)(X)} - \overline{(\underset{V}{L} F)(X)} = N(V, X) + (D_{\bar{V}} A)(X)$$
$$+ A((D_X F)(V)) - \overline{A(D_X V)} - \overline{(D_V A)(X)}. \qquad \text{...(4.10)}$$

PROOF: Barring V in (4.4b) and using (1.1) and (1.2), we get

$$(\underset{\bar{V}}{L} A)(X) = (D_{\bar{V}} A)(X) + A((D_X F)(Y)). \qquad \text{...(4.11}a\text{)}$$

Barring (4.4a) and (4.4b), we get

$$\overline{(\underset{V}{L} F)(X)} = \overline{[V, \bar{X}]} - \overline{[V, X]} \qquad \text{...(4.11}b\text{)}$$

and

$$\overline{(\underset{V}{L} A)(X)} = \overline{(D_V A)(X)} + \overline{A(D_X V)} \qquad \text{...(4.11}c\text{)}$$

respectively.

Adding (4.7a) and (4.11a) and subtracting (4.11b) and (4.11c) from it and using (3.1), we get (4.10).

Corollary 4.2—If V is contravariant almost analytic vector in an almost Sasakian manifold V_n, we have

$$N(V, X) = \overline{(D_V A)(X)} + \overline{A(D_X V)} - (D_{\overline{V}} A)(X) - A((D_X F)(Y)). \quad \ldots(4.12)$$

PROOF: If V is an almost analytic vector, we have

$$(\underset{V}{L} F)(X) = 0 \text{ and } (\underset{V}{L} A)(X) = 0.$$

Above two equations imply

$$(\underset{\overline{V}}{L} F)(X) = 0, \quad (\underset{\overline{V}}{L} A)(X) = 0$$

and

$$\overline{(\underset{V}{L} F)(X)} = 0, \quad \overline{(\underset{V}{L} A)(X)} = 0.$$

Using these in (4.10), we get (4.12).

Theorem 4.6—If V is strictly almost analytic vector, then T is also almost contravariant analytic.

PROOF: If V is strictly almost contravariant analytic, then we have (4.1*a*, *b*) and (4.2*a*, *b*).

Barring V in (4.2*a*) and using (1.1), we get

$$0 = (\underset{(-V+A(V)T)}{L} F \rightarrow)(X),$$

$$= -(\underset{V}{L} F)(X) + A(V)(\underset{T}{L} F)(X).$$

Using (4.1*a*) in this equation, we get

$$(\underset{T}{L} F)(X) = 0. \quad \ldots(4.13a)$$

Similarly, barring V in (4.2*b*) and using (1.1), we get

$$0 = -(\underset{V}{L} A)(X) + A(V)(\underset{T}{L} A)(X) = 0.$$

Again using (4.1*b*) in this equation, we get

$$(\underset{T}{L} A)(X) = 0. \quad \ldots(4.13b)$$

Equations (4.13*a*) and (4.13*b*) prove the required statement.

Theorem 4.7—If X is contravariant almost analytic vector in an almost Sasakian manifold, we have

$$N(X, Y) + A(Y)[X, T] + ((D_X A)(Y) + A(D_Y X))T = 0. \quad \ldots(4.14)$$

PROOF: If X is contravariant almost analytic vector, then we have

$$(\underset{X}{L} F)(Y) = 0 \text{ and } (b)\ (\underset{X}{L} A)(Y) = 0. \qquad \ldots(4.15a)$$

Barring X and Y alternately in (4.15*a*), we get

$$(\underset{\bar{X}}{L} F)(Y) = 0, \quad (\underset{X}{L} F)(\bar{Y}) = 0.$$

Using these equations in (3.5), we get (4.14).

Theorem 4.8—If X and Y are contravariant almost analytic vectors in an almost Sasakian manifold, we have

$$P(X, Y) = A\left(D_{\bar{Y}} X - D_{\bar{X}} Y + (D_Y F)(X) - (D_X F)(Y)\right). \qquad \ldots(4.16)$$

PROOF: (4.16) follows from (4.1) and (3.13).

Theorem 4.9—If T is contravariant almost analytic vector in V_n, we have

$$P(X, T) = A\left((D_Y F)(X) - D_{\bar{X}} Y - (D_X F)(Y)\right) - (\underset{\bar{X}}{L} A)(Y). \qquad \ldots(4.17)$$

PROOF: Proof follows from (4.1) and (4.13).

Theorem 4.10—If T is contravariant almost analytic vector in an almost Sasakian manifold V_n, we have

$$Q(X) = 0. \qquad \ldots(4.18)$$

PROOF: (4.18) is obtained from (3.16) and (4.1).

Theorem 4.11—If T is contravariant almost analytic in an almost Sasakian manifold, we have

$$R(X) = 0. \qquad \ldots(4.19)$$

PROOF: It follows from (3.17) and (4.1).

Theorem 4.12—If X is contravariant almost analytic in an almost Sasakian manifold, we have

$$M(X, Y) = M(Y, X). \qquad \ldots(4.20)$$

PROOF: Using (4.1) in (3.20), we get (4.20).

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