ON THE DEGREE OF APPROXIMATION OF CERTAIN FUNCTIONS BY TRIANGULAR MEANS

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In this paper the author has determined the degree of approximation of certain functions by triangular means in terms of the modulus of continuity. The result obtained generalizes the result of Zygmund (1945).

1. Let

$$\tfrac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.1}$$

and

$$\sum_{\nu=1}^{\infty} (b_\nu \cos \nu x - a_\nu \sin \nu x) \tag{1.2}$$

respectively denote the Fourier series and conjugate Fourier series of a 2π-periodic and Lebesgue integrable function $f(x)$. We shall write

$$S_n(x) = \tfrac{1}{2} a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.3}$$

$$\overline{S_n(x)} = \sum_{\nu=1}^{n} (b_\nu \cos \nu x - a_\nu \sin \nu x)$$

we define

$$t_n(x) = \sum_{k=0}^{n} \Delta\lambda_{n,k} S_k(x) \tag{1.4}$$

$$\overline{t_n(x)} = \sum_{k=0}^{n} \Delta\lambda_{n,k} \bar{S}_k(x)$$

and respectively call these as triangular means of the Fourier series of f and of conjugate Fourier series of f.

2. Zygmund (1945) proved the following theorem:

Theorem A—Suppose that $f(x)$ is periodic, continuous, and that the Fourier series of f is of power series type

$$f(x) \sim \sum_{\nu=0}^{\infty} C_\nu e^{i\nu x}$$

then

$$| \sigma_{n-1}(x) - f(x) | \leq Aw(2\pi/n)$$

where $w(\delta)$ is the modulus of continuity of f and A is an absolute constant.

In this paper we extend Theorem A to the triangular means which includes $(C, 1)$-means as a special case.

3. In what follows we establish the following theorem:

Theorem 3.1—Suppose that $f(x)$ is periodic, continuous and that the Fourier series of f is of power series type

$$f(x) \sim \sum_{\nu=0}^{\infty} C_\nu e^{i\nu x}.$$

Then

$$| t_{n-1}(x) - f(x) | \leq Aw(2\pi/n).$$

4. The proof of the theorem is based on the following lemmas:

Lemma 4.1—Suppose that

$$g(x) \sim \sum_{k=-\infty}^{\infty} k e^{ikx} \tag{4.1}$$

and that

$$| g(x+h) - g(x) | \leq M | h |$$

then

$$| \bar{t}_{n-1}(x) - \bar{g}(x) | \leq \frac{BM}{n},$$

where $\bar{g}(x)$ is the function conjugate to $g(x)$ and $\bar{t}_n(x)$ are the triangular means of the series (1.2).

Lemma 4.2 (Kishore 1971)—If $\{\Delta\lambda_{n,k}\}^n_{k=0}$ is non-negative and non-decreasing sequence with respect to k for $o \leq a < b \leq \infty$, $o \leq t \leq \pi$ and for every n

$$| \sum_{k=a}^{b} \Delta\lambda_{n,n-k} e^{i(n-k)t} | < B t^{-1} \Delta \lambda_{n,n-\tau}$$

where τ is the integral part of $1/t$.

PROOF OF LEMMA 4.1—Since

$$\bar{S}_k(x) - g(x) = \frac{1}{\pi} \int_0^\pi [g(x+t) - g(x-t)] \left\{ \frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right\} dt$$

we see that

$$\bar{t}_n(x) - \bar{g}(x) = \sum_{k=0}^{n} \Delta\lambda_{n,k} \{\bar{S}_k(x) - \bar{g}(x)\} + o(1)$$

$$= \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{1}{\pi} \int_0^{\pi} \{g(x+t) - g(x-t\}.$$

$$\left\{\frac{\sin kt}{kt^2} - \frac{\cos kt}{t}\right\} dt + o(1)$$

$$= \int_0^{\pi} \{g(x+t) - g(x-t)\} \frac{1}{\pi}$$

$$\times \sum_{k=0}^{n} \Delta\lambda_{n,k} \left\{\frac{\sin kt}{kt^2} - \frac{\cos kt}{t}\right\} dt + o(1)$$

$$= \frac{1}{\pi} \int_0^{\pi} [g(x+t) - g(x-t)] J_n(t)\, dt + o(1) \text{ say}$$

where

$$J_n(t) = \sum_{k=0}^{n} \Delta\lambda_{n,k} \left\{\frac{\sin kt}{kt^2} - \frac{\cos kt}{t}\right\} dt.$$

Therefore

$$\bar{t}_n(x) - \bar{g}(x) = \frac{1}{\pi} \left\{ \int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right\} J_n(t)\, dt + o(1)$$

$$= P_n + Q_n \text{ (say)}.$$

For calculating P_n we observe that

$$|J_n(x)| = \frac{1}{\pi} \sum_{k=0}^{n} |\Delta\lambda_{n,k}| Mkt^2$$

uniformly in $o < t \leq 1/n$

$$= \frac{M'n}{\pi}.$$

Therefore

$$|P_n| \leq \int_0^{\pi/n} |g(x+t) - g(x-t)| \frac{M'n}{\pi} dt$$

$$\leq \frac{2Mn}{\pi} \left[\frac{t^2}{2} \right]_0^{\pi/n}$$

$$= M \frac{\pi}{n}.$$

In order to estimate Q_n, we notice that

$$|J_n(x)| = \frac{1}{\pi} \left\{ \left| \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin kt}{kt^2} + \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\cos kt}{t} \right| \right\}$$

$$\leq \frac{1}{\pi} \left\{ \left| \sum_{k=0}^{n} \Delta\lambda_{n,n-k} \frac{\sin (n-k)t}{(n-k)t^2} \right| + \left| \sum_{k=0}^{n} \Delta\lambda_{n,n-k} \frac{\cos (n-k)t}{t} \right| \right\}$$

$$= \frac{1}{\pi} \left\{ \left| \sum_{k=0}^{n} \Delta\lambda_{n,n-k} \frac{\sin (n-k)t}{(n-k)t^2} + M\left(\frac{\Delta\lambda_{n,n-\tau}}{t} \right) \right| \right\}$$

(by Lemma 4.2).

Abel's transformation then gives

$$|J_n(t)| = M_1 \left(\frac{1}{t} \left| \sum_{k=0}^{\tau} \Delta\lambda_{n,n-k} \frac{\sin (n-k)t}{(n-k)t} \right| \right)$$

$$+ M_2 \left(\left| \frac{1}{t} \sum_{k=\tau+1}^{n-1} \Delta\lambda_{n,n-k} \frac{\sin (n-k)t}{(n-k)t} \right| \right) + M\left(\frac{\Delta\lambda_{n,n-\tau}}{t} \right),$$

$$= M_1 \left(\frac{1}{t} \sum_{k=0}^{\tau} \Delta\lambda_{n,n-k} \right) + M_2 \left(\frac{1}{t} \sum_{k=\tau+1}^{n-2} \frac{1}{t} |\Delta^2 \lambda_{n,n-k}| \right)$$

$$+ M_3 \left(\frac{1}{t^2} \Delta\lambda_{n,1} \right) + M_4 \left(\frac{1}{t^2} \Delta\lambda_{n,n-(\tau+1)} \right) + M\left(\frac{\Delta\lambda_{n,n-\tau}}{t} \right),$$

$$= M\left(\frac{\Delta\lambda_{n,n-\tau}}{t} \right).$$

Therefore

$$| Q_n | \leq \frac{M}{\pi} \int_{\pi/n}^{\pi} | g(x+t) - g(x-t) | \frac{\Delta\lambda_{n, n-\tau}}{t} \, dt$$

$$\leq \frac{M}{\pi} \int_{\pi/n}^{\pi} | g(x+t) - g(x-t) | \frac{1}{nt} \, dt \text{ (by lemma 4.2)}$$

$$= \frac{M}{\pi} \int_{\pi/n}^{\pi} \frac{M't}{nt} \, dt$$

$$= \frac{M}{\pi n} \int_{\pi/n}^{\pi} dt$$

$$= \frac{M}{n} - \frac{M}{n^2}$$

$$< \frac{M}{n} .$$

Hence

$$\bar{t}_n(x) - \bar{g}(x) \leq \frac{M\pi}{n} + \frac{M}{n}$$

$$= \frac{M}{n}(\pi + 1)$$

and

$$\bar{t}_{n-1}(x) - \bar{g}(x) \leq \frac{BM}{n} \quad \text{where } B = \pi + 1$$

which completes the proof of the lemma.

5. *Proof of Theorem* 3.1—Suppose that the Fourier series of $f(x)$ is of power series type, so that

$$\bar{f} = -if.$$

If

$$|f(x+h) - f(x)| \leq M |h|$$

then

$$| t_{n-1}(x) - f(x) | = | \bar{t}_{n-1}(x) - \bar{f}(x) | \leq B \frac{M}{n} .$$

Now, in order to complete the proof, we introduce a function

$$f_h(x) = \frac{1}{h} \int_{-h}^{h} f(x + t)\, dt = \frac{F(x+h) - F(x-h)}{2h}$$

$$\sim \sum_{k=0}^{\infty} C_k\, e^{ikx} \left(\frac{\sin kh}{kh}\right),$$

where $F(x)$ is primitive of $f(x)$. Hence df_h/dx exists, is continuous and does not exceed $w(2h)/2h$ [$\leq w(h)/h$] in absolute values. Moreover, the Fourier series of f_h is also of power series type. Now

$$\left| t_{n-1}(x,f) - f(x) \right| \leq \left| t_{n-1}(x,f) - t_{n-1}(x,f_h) \right| + \left| t_{n-1}(x,f_h) - f_h(x) \right|$$
$$+ \left| f_h(x) - f(x) \right|,$$
$$\leq \left| A_n \right| + \left| B_n \right| + \left| C_n \right| \text{ (say)}.$$

Here we observe that

$$\left| C_n \right| = \left| \frac{1}{2h} \int_{-h}^{h} [f(x+t) - f(t)]\, dt \right|$$
$$\leq w(h)$$
$$\left| B_n \right| \leq \frac{B\, w(h)}{h} \,.\, 1/n$$

and

$$\left| A_n \right| \leq \left| t_{n-1}(x, f - f(x) \right| \leq \| f - f_h \| \leq w(h).$$

Therefore

$$\left| t_{n-1}(x,f) - f(x) \right| = \left| t_{n-1}(x) - f(x) \right|$$
$$\leq w(h) + B\frac{w(h)}{h} \,.\, \frac{1}{n} + w(h)$$
$$\leq w(h)\left[2 + \frac{B}{hn}\right].$$

Putting

$h = \frac{2\pi}{n}$, we get

$$| t_{n-1}(x) - f(x) | \leq w\left(\frac{2\pi}{n}\right) \left[2 + \frac{B}{2\pi}\right]$$

$$= Aw\left(\frac{2\pi}{n}\right) \text{ where } A = 2 + \frac{B}{2\pi}.$$

This completes the proof of the theorem.

REFERENCES

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