ON THE APPROXIMATION OF FUNCTIONS BY GENERAL ORTHONORMAL POLYNOMIALS ON A FINITE SEGMENT

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Suetin (1964) has proved that if $f(x)$ has r continuous derivatives on $[-1, 1]$ and if $f^{(r)}(x) \in \text{Lip}\ \alpha$, then

$$\left| f(x) - \sum_{k=0}^{n} a_k P_k(x) \right| \leq \frac{C \log n}{n^{r+\alpha} - \frac{1}{2}},\ x \in [-1, 1].$$

for $r+\alpha > 1/2$, where $P_n(x)$ is the normalized Legendre polynomial in $[-1, 1]$. Considering the Fourier series of general orthonormal polynomials, we have generalized the above theorem.

§ 1.1. An orthonormal set of functions $w_0(x)$, $w_1(x)$, ..., $w_l(x)$, l finite or infinite with weight function $p(x)$ is defined by the relation

$$\int_a^b w_n(x)\, w_m(x)\, p(x)\, dx = \delta_{nm},\ m, n = 0, 1, 2, \ldots$$

where

$$\delta_{nm} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

Let $\{w_n(x)\}$ be a given orthonormal set, finite or infinite. To an arbitrary real valued function $f(x)$, let there correspond the formal Fourier expansion

$$f(x) \sim C_0 w_0(x) + C_1 w_1(x) + \ldots + C_n w_n(x) + \ldots$$

The coefficients C_n, called the Fourier coefficients of $f(x)$ with respect to the given system, are defined by

$$C_k = \int_a^b f(x)\, w_n(x)\, p(x)\, dx,\ n = 0, 1, 2, \qquad \ldots(1.1.1)$$

and it exists if $f(x) \in L^2_{p(x)}[a, b]$, i.e. $p(x) f^2(x)$ is summable on the segment $[a, b]$.

§ 1.2. It is well known (Natanson 1949) that if the function $f(x)$ on the segment $[a, b]$ has r continuous derivatives, and the modulus of continuity of $f^{(r)}(x)$ is $\psi_r(\delta)$, then there exists an algebraic polynomial $\rho_n(x)$ of degree less than or equal to n, such that for $n > r$,

$$| f(x) - \rho_n(x) | \leq \frac{C_r (b-a)^r}{n^r} \psi_r \left(\frac{b-a}{2(n-r)} \right) \qquad \ldots(1.1.2)$$

where C_r is a constant depends only r.

§ 2.1. Suetin (1964) has proved the following result in a Fourier series of Legendre polynomials $P_n(x)$; normalized in $[-1, 1]$.

Theorem A—If $f(x)$ has r continuous derivatives on $[-1, 1]$ and $f^{(r)}(x) \in$ Lip α, then

$$\left| f(x) - \sum_{k=0}^{n} a_k P_k(x) \right| \leq \frac{C \log n}{n^{r+\alpha} - 1/2}, \quad x \in [-1, 1]$$

for $r + \alpha > 1/2$.

§ 3. In the present note we extend the above result by considering the Fourier series of general orthonormal polynomials. In fact, we prove the following result:

Theorem 3.1—If the function $f(x)$ on the segment $[a, b]$ has r continuous derivatives with the modulus of continuity $\psi_r(\delta)$, of $f^{(r)}(x)$ and $w_n(t)$ are orthonormal polynomials in $[a, b]$ so that

$$| w_n(x) | \leq A^* \lambda_n \text{ for } x \in [a, b] \qquad ...(3.1.1)$$

λ_n is a positive increasing function of n,

$$p(t) \, | w_n(t) | \leq A \text{ for } t \in [a, b] \qquad ...(3.1.2)$$

then

$$| f(x) - S_n(x) | \leq \frac{A \lambda_n \log n}{n^r} \psi_r \left(\frac{b-a}{2(n-r)} \right) \qquad ...(3.1.3)$$

where

$$S_n(x) = \sum_{k=0}^{n} C_k w_k(x)$$

and C_k is as defined in (1.1.1).

PROOF OF THEOREM: For $a \leq x \leq b$ we write

$$| f(x) - S_n(x) | \leq | f(x) - \rho_n(x) | + | \rho_n(x) - S_n(x) |$$

$$\leq | f(x) - \rho_n(x) + \int_a^b p(t) \, | \rho_n(t) - f(t) | \, \left| \sum_{k=0}^{n} w_k(t) \, w_k(x) \right| dt$$

$$= I_1 + I_2,$$

where $\rho_n(x)$ is defined in (1.1.2) an algebraic polynomial of degree less than or equal to n.

$$I_1 = | f(x) - \rho_n(x) |$$

$$\leq \frac{C_r (b-a)^r}{n^r} \psi_r \left(\frac{(b-a)}{2(n-r)} \right).$$

A^* is a constant, not necessarily the same at each occurrence.

Therefore

$$I_1 \leq \frac{B}{n^r}\, \psi_r\left(\frac{b-a}{2(n-r)}\right). \qquad \text{...(3.1.4)}$$

We denote by $\Delta_n(x)$ the part of $[a, b]$ on which $|x-t| \leq 1/n$ and by $\delta_n(x)$ the rest of this interval. In order to establish I_2 we write

$$I_2 = \int_a^b p(t)\,|\rho_n(t) - f(t)|\;\Big|\sum_{k=0}^{n} w_k(t)\, w_k(x)\Big|\, dt$$

$$= \int_{\Delta n(x)} + \int_{\delta_n(x)} = I^*_1 + I_2^*$$

$$I_1^* = \int_{\Delta n(x)} p(t)\,|\rho_n(t) - f(t)|\;\Big|\sum_{k=0}^{n} w_k(t)\, w_k(x)\Big|\, dt$$

$$= \frac{C_r(b-a)^r}{n^r}\, \psi_r\left(\frac{b-a}{2(n-r)}\right) \int_{\Delta n(x)} p(t) \Big|\sum_{k=0}^{n} w_k(t)\, w_k(x)\Big|\, dt$$

$$\leq \frac{C_r(b-a)^r}{n^r}\, \psi_r\left(\frac{b-a}{2(n-r)}\right) \int_{\Delta_n(x)} p(t) \sum_{k=0}^{n} |w_k(t)|\,|w_k(x)|\, dt$$

$$I_1^* \leq \frac{B'}{n^r}\, \psi_r\left(\frac{b-a}{2(n-r)}\right) \frac{1}{n} \sum_{k=0}^{n} \lambda_k. \qquad \text{...(3.1.5)}$$

Further to find the estimate over $\delta_n(x)$ we use the following Christoffel's formula:

$$\sum_{k=0}^{n} w_k(t)\, w_k(x) = \theta_n\, \frac{w_{n+1}(x)\, w_n(t) - w_n(x)\, w_{n+1}(t)}{x-t}$$

for $0 < \theta_n \leq 1$.

Since $|x-t| > 1/n$ for $t \in \delta_n(x)$, hence by (3.1.1) and (3.1.2) we have,

$$I_2^* = \int_{\delta_n(x)} |\rho_n(t) - f(t)|\,.\,p(t) \left|\frac{w_{n+1}(x)\, w_n(t) - w_n(x)\, w_{n+1}(t)}{x-t}\right| dt$$

$$\leq \frac{C'}{n^r}\, \psi_r\left(\frac{b-a}{2(n-r)}\right) \int_{\delta_n(t)} p(t)\, \frac{|w_{n+1}(x)\, w_n(t) - w_n(x)\, w_{n+1}(t)|}{|x-t|}$$

$$\leq \frac{C'\lambda_n}{n^r}\, \psi_r\left(\frac{b-a}{2(n-r)}\right) \int_{\delta_n(x)} p(t)\Big[\,|w_n(t)| + |w_{n+1}(t)|\,\Big] \frac{dt}{|x-t|}$$

$$\leq \frac{C'' \lambda_n}{n^r} \psi_r \left(\frac{b-a}{2(n-r)}\right) \int_{\delta_n(x)} \frac{dt}{|x-t|}$$

$$\leq \frac{C'' \lambda_n}{n^r} \psi_r \left(\frac{b-a}{2(n-r)}\right) \log n. \qquad ...(3.1.6)$$

Hence from (3.1.4), (3.1.5) and (3.1.6) it follows that

$$|f(x) - S_n(x)| \leq \frac{B}{n^r} \psi_r \left(\frac{b-a}{2(n-r)}\right) + \frac{B'}{n^r} \psi_r \left(\frac{b-a}{2(n-r)}\right)$$

$$\times \frac{1}{n} \sum_{k=0}^{n} \lambda_k + \frac{C'' \lambda_n}{n^r} \psi_r \left(\frac{b-a}{2(n-r)}\right) \log n.$$

$$|f(x) - S_n(x)| \leq \frac{A \lambda_n \log n}{n^r} \psi_r \left(\frac{b-a}{2n-r}\right).$$

This completes the proof of Theorem 3.1.

In fact our theorem generalizes Theorem A of Suetin (1964), in the sense that we have determined the approximation of function by general orthonormal polynomials and for Legendre polynomials. It may also be remarked that by giving different values of λ_n, $p(x)$ in (3.1.1), (3.1.2) and (3.1.3) our theorem reduces to different results for normalized Jacobi polynomials, Chebyshev polynomial of first and second kind, etc., i.e., if we choose

(i) $\lambda_n = 1$

$p(x) = 1$

our theorem reduces for the systems

$$\sqrt{\frac{2}{\pi}}, \sqrt{\frac{2}{\pi}} \cos x, \sqrt{\frac{2}{\pi}} \cos ex...$$

$$\sqrt{\frac{2}{\pi}} \sin x, \sqrt{\frac{2}{\pi}} \sin 2x \ ...$$

which is orthonormal on $[0, \pi]$.

(ii) $\lambda_n = C\sqrt{n}$

$p(t) = (1 - t^2)^{1/4}, \ t \in [-1, 1]$

our theorem gives a theorem on normalized Legendre polynomial $P_n(x)$.

(iii) $\lambda_n = \dfrac{2n+1}{\sqrt{\pi}}$

$p(t) = (1 - t^2)^{1/2}, \ t \in (-1, 1]$

in our theorem we get a result for normalized Jacobi polynomials

$$P_n^{\alpha,\beta}(x), \text{ where } \alpha = \tfrac{1}{2},\ \beta = -\tfrac{1}{2} \text{ and } p(x) = (1-x)^\alpha (1+x)^\beta, \alpha > -1,$$
$$\beta > -1.$$

(iv) $\lambda_n = Dn^\alpha$

$$p(t) = (1-t^2)^\alpha,\ t \in [-1,\ 1]$$

in our result, then we get the same result for normalized Jacobi polynomial $P_n^{(\alpha,\beta)}$ when $\alpha = \beta$.

(v) We have the results for normalized Chebyshev polynomials of first and second kind

$$T_n(x) = \cos n\theta,\ x = \cos\ \theta \text{ for } \alpha = \beta = -1/2$$

and

$$U_n(x) = \sin\ (n+1)\,\theta/\sin\ \theta,\ x = \cos\ \theta \text{ for } \alpha = \beta = 1/2$$

for which if we take

$$\lambda_n = 1$$
$$p(t) = (1-t^2)$$
$$\lambda_n = \frac{2}{\pi}(n+1)$$
$$p(t) = (1-t^2)^{1/2}\quad t \in [-1,\ 1]$$

in our theorem respectively, then it becomes the direct consequence of our theorem.

References

Natanson, I. P. (1949). Constructive Theory of Functions, State Publishing House of Technical-Theoretical Literature, Moscow, Leningrad, pp. 122–23, 263–64.

Suetin, P. K. (1964). Representation of continuous and differentiable functions by Fourier series of Legendre polynomials. *Soviet Math. Dock.*, pp. 1408–10.